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The Neumann problem for one-dimensional parabolic equations with linear growth Lagrangian: evolution of singularities

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Abstract. In this paper, we obtain existence and uniqueness of strong solutions to the inhomogenous Neumann initial-boundary problem for a parabolic PDE which arises as a generalization of the time-dependent minimal surface equation. Existence and regularity in time of the solution are proved by means of a suitable pseudoparabolic relaxed approximation of the equation and the corresponding passage to the limit. Our main result is monotonicity in time of the positive and negative singular parts of the distributional space derivative for bounded variation initial data. Sufficient conditions for instantaneous L^1-$W^{1,1}_{\mathrm{loc}}$ or BV-$W^{1,1}_{\mathrm{loc}}$ regularizing effects are also discussed.

1. Introduction

In this paper we consider the following one-dimensional inhomogeneous Neumann initial-boundary problem

$$(P^{u_0}_{\alpha,\beta})\qquad \begin{cases} u_t = [\mathbf{a}(u,u_x)]_x + F & \text{in } Q_{0,T} := (0,T)\times\Omega,\\ \mathbf{a}(u(0,t),u_x(0,t)) = \alpha, & \text{for } t\in(0,T),\\ \mathbf{a}(u(L,t),u_x(L,t)) = \beta, & \text{for } t\in(0,T),\\ u(\cdot,0) = u_0 & \text{in } \Omega\,, \end{cases}$$

where $\Omega = (0,L)\subset\mathbb{R}$, $T>0$, $u_0\in L^2(\Omega)$ and $F\in C^2(\overline{Q}_{0,T})$.
The following assumptions on the flux will be always made throughout the paper:
(H_1) $\mathbf{a}:\mathbb{R}^2\to\mathbb{R}$ is Lipschitz continuous,

$$\mathbf{a}(z,0)=0,\qquad |\mathbf{a}(z,\xi)|\le 1 \quad \text{for all } (z,\xi)\,, \tag{1.1}$$

and there exists $f:\mathbb{R}^2\to\mathbb{R}$, $f\in C^1(\mathbb{R}^2)$, convex with respect to ξ, such that

$$\mathbf{a}(z,\xi)=\partial_\xi f(z,\xi),\qquad |\partial_z f(z,\xi)|\le\gamma \quad \text{for all } (z,\xi)\in\mathbb{R}^2\,,\quad \text{for some } \gamma>0. \tag{1.2}$$

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(H_2) There exist $C_0,\ D_0,\ C_1 > 0$ such that

$$\max\{|\alpha|, |\beta|\} \leq C_0 \leq 1, \quad C_0|\xi| - D_0 \leq f(z,\xi) \leq C_1\big(|\xi| + |z| + 1\big) \tag{1.3}$$

for every $(z,\xi) \in \mathbb{R}^2$; moreover,

$$f^0(z,\xi) := \lim_{t\to 0^+} tf\left(z, \frac{\xi}{t}\right) = |\xi| \qquad (z,\ \xi \in \mathbb{R}). \tag{1.4}$$

As a consequence of (H_1), for every $z \in \mathbb{R}$ the mapping $\xi \to \mathbf{a}(z,\xi)$ is nondecreasing in $\mathbb{R}$ and the function $h : \mathbb{R}^2 \to \mathbb{R}$,

$$h(z,\xi) := \mathbf{a}(z,\xi)\,\xi\,, \tag{1.5}$$

satisfies $h(z,\xi) \geq 0$ for all $(z,\xi) \in \mathbb{R}^2$. In addition, by (1.1), (1.3) and the convexity of f, we also get

$$\mathbf{a}(z,\xi)\,\xi \geq \mathbf{a}(z,\xi)\,\eta + f(z,\xi) - f(z,\eta)\,, \tag{1.6}$$

$$C_0|\xi| - C_1(|z| + 1) - D_0 \leq f(z,\xi) - f(z,0) \leq h(z,\xi) \leq |\xi|\,. \tag{1.7}$$

In view of (1.1), (1.4) and (1.6), we have

$$h^0(z,\xi) := \lim_{t\to 0^+} t\,h\left(z, \frac{\xi}{t}\right) = |\xi| \qquad (z, \xi \in \mathbb{R}),$$

$$\lim_{\xi\to+\infty} \mathbf{a}(z,\xi) = 1, \qquad \lim_{\xi\to-\infty} \mathbf{a}(z,\xi) = -1 \qquad \text{for all } z \in \mathbb{R}\,. \tag{1.8}$$

The parabolic PDE in $(P^{u_0}_{\alpha,\beta})$ includes the case of the time-dependent minimal surface equation for which $\mathbf{a}(z,\xi) = \mathbf{a}(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$. In this case, the Lagrangian corresponds to the nonparametric area integrand; i.e.

$$f = f(\xi) = \sqrt{1 + |\xi|^2},$$

which arises in several models of physical systems describing, for instance, the motion of capillary surfaces or the motion of grain boundaries (see [14]).

The time-dependent minimal surfaces equation has been widely studied since the pioneering paper by Lichnewski and Temam [21] in which the authors proved existence and uniqueness of "pseudo-solutions" for the corresponding Dirichlet problem (in a multidimensional domain).

Several generalizations are available in the literature, both for the inhomogenous Dirichlet problem and for the homogenous Neumann one. In [5,6], F. Andreu, V. Caselles and J. Mazón obtained existence and uniqueness of entropy solutions for general Lagrangians with linear growth with respect to the gradient with similar hypothesis to ours. In a previous work [22], we proved that the homogenous Neumann problem for $(P^{u_0}_{\alpha,\beta})$ (i.e., $\alpha = \beta = 0$) has a unique strong solution for any u_0 in $L^2(\Omega)$ in any dimension.

On the other hand, the inhomogeneous Neumann problem for this type of parabolic equations is much less studied. To the best of our knowledge, even for the case of linear growth functionals, as in the case of the time-dependent minimal surfaces problems, the available results are limited to the study of minimizers and their properties (see [8]) and not to the evolution in time. The only result in this direction in the parabolic case is the existence and uniqueness of entropy solutions to a mixed Neumann–Dirichlet problem for the relativistic heat equation, also in the one-dimensional case ([4]). We note that the entropy solutions constructed in [4] do not satisfy any reasonable type of time regularity. In fact, the distributional time derivative of the solutions might not be even a Radon measure.

Our purpose in this paper is twofold. First of all, to generalize the results in [22] to the nonhomogenous Neumann case; i.e. to obtain existence and uniqueness of strong solutions to $(P^{u_0}_{\alpha,\beta})$.

Concerning existence, we follow the strategy in [22]; i.e. we consider the following *relaxed-pseudoparabolic regularization* for problem $(P^{u_0}_{\alpha,\beta})$

$$u_t = (\mathbf{a}(u, u_x) + \sqrt{\epsilon} u_x)_x + \epsilon u_{txx} + F \,,$$

coupled with the corresponding Neumann boundary conditions. We show that they form a family of well-posed approximating problems $(P^{u_0}_{\alpha,\beta})_\epsilon$. Coupled with good a priori estimates, we are able to deduce the existence of strong solutions to problem $(P^{u_0}_{\alpha,\beta})$ for any $u_0 \in L^2(\Omega)$. On the other hand, uniqueness will be proved thanks to the monotonicity of the diffusion, via a comparison theorem between sub- and supersolutions in $L^1(\Omega)$.

Our main objective is, however, to show that, under suitable conditions on the Lagrangian, in the case that the initial datum is a bounded variation function, the singular part of the positive and negative part of the space distributional derivative of the solutions $D_\pm u(t)$ is controlled by that of the initial datum, $D^s_\pm(u_0)$. In more detail, besides (H_1)–(H_2), we shall consider in Theorem 3.6 below the following assumption:

(H_3) $\mathbf{a}(z, \xi) \in C^1(\mathbb{R}^2)$ and the map $\xi \mapsto \mathbf{a}(z, \xi)$ is strictly increasing for every $z \in \mathbb{R}$; moreover, there exists $\sigma_0 > 1$ such that for every $M > 0$, there exists $D_M > 0$ such that

$$|\partial_z \mathbf{a}(z, \xi)| \le \frac{D_M}{(1 + |\xi|)^{\sigma_0}} \quad \text{for all } (z, \xi) \in \mathbb{R}^2,\ |z| \le M \,. \tag{1.9}$$

There exist $0 < \sigma_1, \sigma_2 \le \sigma_0$ such that $\sigma_i < 2\sigma_0 - 1$, for $i = 1, 2$ and, for every $M > 0$, there exist $C^\pm_{i,M} > 0$ and $\varphi_{i,M} \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $(i = 1, 2)$ such that $C^+_{i,M} < \varphi_{i,M}(z) < C^-_{i,M}$ for $|z| \le M$ and

$$-\frac{C^-_{1,M}}{\varphi_{1,M}(z) + |\xi|^{\sigma_1}} \le \mathbf{a}(z, \xi) - 1 \le -\frac{C^+_{1,M}}{\varphi_{1,M}(z) + |\xi|^{\sigma_1}} \quad \text{for all } |z| \le M,\ \xi > 0 \,, \tag{1.10}$$

$$\frac{C^{+}_{2,M}}{\varphi_{2,M}(z)+|\xi|^{\sigma_2}} \le \mathbf{a}(z,\xi)+1 \le \frac{C^{-}_{2,M}}{\varphi_{2,M}(z)+|\xi|^{\sigma_2}} \quad \text{for all } |z|\le M,\ \xi<0\,. \tag{1.11}$$

Let us explicitly observe that:

- (H_3) is satisfied with $\sigma_0=\sigma_1=\sigma_2=\sigma\ge 2$, for the following class of nonlinearities

$$\mathbf{a}(z,\xi) := \frac{|\xi|^{\sigma-1}\mathrm{sgn}(\xi)}{(\varphi(z)+|\xi|^{\sigma})^{1-1/\sigma}} \quad \text{or} \tag{1.12}$$

$$\mathbf{a}(z,\xi) := \frac{\xi}{(\varphi(z)+|\xi|^{\sigma})^{1/\sigma}} \tag{1.13}$$

  with $\varphi\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, $\varphi(z)\ge c>0$ for all $z\in\mathbb{R}$ (see Remark 3.2 below).
- In the case $F\equiv 0$, the condition $\sigma_i\le\sigma_0$ is not necessary for the results in Theorem 3.6 to hold. In addition, if the flux $\mathbf{a}(z,\xi)\equiv\mathbf{a}(\xi)$ does not depend explicitly on z, in assumption (H_3) it suffices to require that (1.10) and (1.11) hold true for some choice of $\sigma_1,\ \sigma_2>0$. In the case $\alpha=\beta=0$ and $u_0\in BV(\Omega)$, this is in agreement with the *monotonicity-in-time properties* in [24, Theorem 2.9] (see also [25, Theorem 3.7]) for the singular part of Radon measure-valued solutions to the Dirichlet initial-boundary value problem

$$\begin{cases} v_t=[\mathbf{a}(v)]_{xx} & \text{in } Q_{0,T}\,,\\ \mathbf{a}(v)=0 & \text{on } \partial\Omega\times(0,T)\,,\\ v(0)=\mu & \text{in } \Omega\,, \end{cases} \tag{1.14}$$

  with a finite Radon measure $\mu\in\mathcal{M}(\Omega)$ as initial condition (notice that (1.14) is formally obtained by differentiating with respect to x the equation in $(P^{u_0}_{\alpha,\beta})$, and setting $v:=u_x$, $\mu:=Du_0$).
- In the general case where the nonlinearity $\mathbf{a}(z,\xi)$ depends explicitly on the first variable z, the optimality of the technical conditions relating the coefficients σ_i $(i=0,1,2)$ in assumption (H_3) remains an open problem.

In particular, as a consequence of Theorem 3.6 below, for solutions of $(P^{u_0}_{\alpha,\beta})$ no new jumps are created through the evolution and we observe that the size of jumps cannot increase in time. Moreover, it also leads to a control on the Cantor part of the derivative. We point out that similar results have already been reported for the case of the total variation flow. In particular, in the one-dimensional case, the total variation of the solution can be controlled by that of the initial data in a completely local way (see [20]) while in the multidimensional case, this has only been shown for the jump part (see [17] for dimension less than or equal to 7 or [23] for a related result in any dimension).

The main assumptions on the Lagrangian in (H_3) are regularity, monotonicity and a suitable convergence rate of the mapping $\xi\to\mathbf{a}(z,\xi)$ to the saturation values ± 1

as $\xi \to \pm\infty$. First of all, regularity allows to show that the sequence formed by space derivatives of the approximating solutions and the corresponding vector fields converge a.e., respectively, to the density of the absolutely continuous part of the derivative of the solutions u and the corresponding vector field. On the other hand, finer a priori estimates in the case that $u_0 \in BV(\Omega)$ show that a subsequence of the positive (resp. the negative part) of the derivative of the solutions converge as measures to the positive part (resp. the negative part) of the distributional derivative of the solution. The final step will be to show that $D^s_{\pm}u(t) \leq D^s_{\pm}(u_0)$ holds in the sense of Radon measures. This will be possible thanks to the hypothesis on the rate of convergence towards the saturation.

Finally, we show that whenever the convergence rate of the mappings $\xi \mapsto \mathbf{a}(z, \xi)$ to the limiting values ± 1 is not too fast, *i.e.* if

(H_4) (i) there exist $\sigma \in (0, 1/2]$ and, for every $M > 0$, $C^+_{3,M} > 0$ such that

$$1 - \mathbf{a}(z, \xi) \geq \frac{C^+_{3,M}}{(1 + |\xi|)^\sigma} \quad \text{for all } |z| \leq M, \ \xi > 0\,,$$

(ii) there exist $\sigma \in (0, 1/2]$ and, for every $M > 0$, $C^-_{3,M} > 0$ such that

$$1 + \mathbf{a}(z, \xi) \geq \frac{C^-_{3,M}}{(1 + |\xi|)^\sigma} \quad \text{for all } |z| \leq M, \ \xi < 0\,,$$

solutions to $(P^{u_0}_{\alpha,\beta})$ exhibit an instantaneous $L^1(\Omega)$-$W^{1,1}_{\rm loc}(\Omega)$ regularizing effect. Namely, the singular part of the distributional derivative $Du(t)$ vanishes for all $t > 0$ (see Theorem 3.7 below).

However, we observe that the condition $\sigma \in (0, 1/2]$ in assumption (H_4) does not seem to be sharp. Indeed, in the case where the nonlinearity $\mathbf{a}(z, \xi)$ satisfies assumption (H_3) with $\sigma_1, \sigma_2 \in (0, 1]$, instantaneous $BV(\Omega)$-$W^{1,1}_{\rm loc}(\Omega)$ regularizing effects continue to hold for strong solutions to $(P^{u_0}_{\alpha,\beta})$, for any initial data $u_0 \in BV(\Omega)$ (see Theorem 3.8 below).

For a nonlinearity $\mathbf{a}(z, \xi) \equiv \mathbf{a}(\xi)$ independent of z, this is in agreement with [10, Theorem 2.4], where instantaneous disappearance of jump-discontinuities has been proven for solutions to the Cauchy problem for the equation $u_t = [\mathbf{a}(u_x)]_x$, with bounded and increasing initial data, and suitable assumptions on the flux, which are in particular fulfilled when hypothesis (H_3) holds with $\sigma_i \in (0, 1]$. Similar results concerning the regularity of the solutions to variational linear growth problems have been recently reported in the literature in this case that $\mathbf{a}(z, \xi) = \mathbf{a}(\xi)$ is independent of z. The Dirichlet problem has been widely studied since the work of Bildhauer and Fuchs [11] where a condition of $\mu-$ellipticity on the flux was introduced for Total Variation related minimization problems. We point out that the condition of μ-ellipticity is similar to condition (H_4) with $\sigma = \mu + 1$. This study has been then generalized to the vector valued case [12] or to Dirichlet problems for radially symmetric data [13]. Concerning the Dirichlet problem we also mention the results in [9] about global

Lipschitz minimizers or [15] about boundary regularity. Finally, we mention the regularity of solutions for related parabolic problems for Lipschitz initial data and periodic boundary conditions studied in [16].

We include in the equation a smooth forcing term, F. In a subsequent work, we will use the results in here to study some qualitative properties of the solutions. In particular, we will study finite time vanishing at some point (when $u_0 \geq C > 0$) or infinite time blow up to particular examples of equations satisfying hypothesis $(H_2) - (H_3)$ and a slightly weaker condition than (H_1) (see hypothesis $(H_1)'$ in [22]).

We finish this introduction with the organization of the paper: Sect. 2 includes notations and preliminaries on bounded variation functions that we need in the paper. In Sect. 3 we collect the definitions of sub- and supersolutions and the main results of the paper. Section 4 is devoted to the well posedness of the approximating problems $(P^{u_0}_{\alpha,\beta})_\epsilon$ and to show some inequalities needed for the a priori estimates that we use to pass to the limit. This a priori estimates (both for L^2 and for BV initial data) are obtained in Sect. 5. In Sect. 6, we prove the existence and comparison theorems for solutions. Finally, in Sect. 7 we prove the monotonicity properties of the positive and negative singular parts of the derivative of the solutions in the case of BV initial data under hypothesis (H_3) and we also prove the instantaneous regularizing effect under hypothesis (H_4).

2. Preliminaries

2.1. Radon measures and functions of bounded variation

Throughout the paper, $\Omega = (0, L) \subset \mathbb{R}$ will denote an open bounded interval. For any function $f : \Omega \to \mathbb{R}$, we let f^+, f^- be its positive and negative part; i.e $f^+ := \max\{f, 0\}$, $f^- := \max\{-f, 0\}$. We denote by $\mathcal{M}(\Omega)$ the space of the finite (signed) Radon measures on Ω, and by $\mathcal{M}^+(\Omega)$ the cone of its nonnegative elements. For every $\mu \in \mathcal{M}(\Omega)$ and for every Borel set $E \subseteq \Omega$, the restriction $\mu \llcorner E$ of μ to E is defined by $(\mu \llcorner E)(A) := \mu(E \cap A)$ for every Borel set $A \subseteq \Omega$. Given two nonnegative measures μ and ν, we write $\mu << \nu$ if μ is absolutely continuous with respect to ν (see e.g. [3, Definition 1.24]). Every $\mu \in \mathcal{M}(\Omega)$ has a unique decomposition $\mu = \mu_{ac} + \mu_s$, with $\mu_{ac} = \mu_r \mathcal{L}$ absolutely continuous (here $\mu_r \in L^1(\Omega)$ is the density of μ_{ac}) and μ_s singular with respect to the Lebesgue measure $\mathcal{L}$.

We denote by $BV(\Omega)$ the Banach space of functions of bounded variation in Ω:

$$BV(\Omega) := \left\{ u \in L^1(\Omega) \mid Du \in \mathcal{M}(\Omega) \right\}, \quad \|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|_{\mathcal{M}(\Omega)},$$

where Du is the first-order distributional derivative of u. We say that $u \in BV_{\text{loc}}(\Omega)$ if $u \in BV(\Omega')$ for every open subset $\Omega' \subset\subset \Omega$. We recall that given any $u \in BV(\Omega)$ the measure Du can be decomposed into its absolutely continuous and singular parts,

$$Du = u_x \mathcal{L} + D^s u,$$

u_x being the density of the absolutely continuous part of Du. We use standard notations and results for BV functions as in [3] and we will always identify a BV function with its precise representative. Finally, for a BV-function u, $D_{\pm}u$ will denote the positive (and negative, respectively) part of the measure Du.

The following lemma will be also used below:

Lemma 2.1. *Let $u_0 \in BV(\Omega)$. Then, for any sequence $\epsilon_k \to 0^+$, there exists $\{u_{0k}\} \subseteq H^1(\Omega)$ such that*

$$\begin{cases} (i)\ u_{0k} \to u_0 \ \ \textit{strictly in} \ \ BV(\Omega)\,, \\ (ii)\ \sqrt{\epsilon_k}\displaystyle\int_\Omega \big((u_{0k})_x\big)^2\,\mathrm{d}x \to 0\,, \\ (iii)\ (u_{0k})_x \to (u_0)_x \ \ a.e. \ \ in \ \ \Omega\,, \\ (iv)\ \displaystyle\lim_{k\to\infty}\int_\Omega ((u_{0k})_x)_{\pm}\,\rho\,\mathrm{d}x = \int_\Omega \rho\, dD_{\pm}u_0\,. \end{cases} \quad \text{as } k \to \infty. \tag{2.1}$$

Proof. This Lemma is essentially contained in [22, Lemma 2.1], where a sequence satisfying (i)–(iii) is constructed as follows. Let $T : BV(\Omega) \to BV(\mathbb{R})$ an extension operator and let $\rho_\epsilon(x) := \frac{1}{\epsilon}\rho(x/\epsilon)$ a standard mollifier. Then, $u_{0k} := (Tu_0) \star \rho_{\epsilon_k^{1/4}}$ satisfies (i)–(iii). Moreover, we have that

$$\sup_{k\in\mathbb{N}} \|u_{0kx}\|_{L^1(\Omega)} < \infty, \ \ u_{0kx} \overset{*}{\rightharpoonup} Du_0 \ \text{ in } \ \mathcal{M}(\Omega)\,, \ \ u_{0kx} \to u_{0x} \ \ a.e. \ \text{ in } \ \Omega\,. \tag{2.2}$$

Let $\{\nu_k\} \subseteq \mathcal{Y}(\Omega;\mathbb{R})$ be the sequence of Young measures associated to $\{u_{0kx}\}$. Then there exists a Young measure $\nu \in \mathcal{Y}(\Omega;\mathbb{R})$ such that $\nu_k \to \nu$ narrowly, possibly up to a subsequence ([26]). By (2.2)$_3$, for $a.e.\, x \in \Omega$ the disintegration of ν satisfies $\nu_{(x)} = \delta_{\{u_{0x}(x)\}}$, whence $[u_{0x}]^{\pm} = \int_{\mathbb{R}} \xi^{\pm} d\nu(\xi)$ $a.e.$ in Ω. Then it can be easily checked that $[u_{0kx}]^{\pm} \overset{*}{\rightharpoonup} [u_{0x}]^{\pm}\mathcal{L} + \lambda_{(\pm)}$ in $\mathcal{M}(\Omega)$, where $\lambda_{(\pm)} \in \mathcal{M}^+(\Omega)$ and $\lambda_{(\pm)} \le D^s_{\pm}u_0$ in $\mathcal{M}(\Omega)$, since $[u_{0kx}]^{\pm} \le D_{\pm}u_0 \star \rho_{\epsilon_k}$ for all k. In particular, this implies that the measures $\lambda_{(\pm)}$ are singular with respect to the Lebesgue measure and mutually singular. Therefore, we get that $\lambda_{(\pm)} = D^s_{\pm}u_0$, combining the convergences $u_{0kx} \overset{*}{\rightharpoonup} Du_0 = u_{0x}\mathcal{L} + D^s u_0$ and $u_{0kx} = [u_{0kx}]^+ - [u_{0kx}]^- \overset{*}{\rightharpoonup} u_{0x}\mathcal{L} + \lambda_{(+)} - \lambda_{(-)}$, and using the uniqueness of the Lebesgue and Jordan decomposition of the measure Du_0. Then (iv) follows at once. □

For every $u \in BV(\Omega)$ and $v \in H^1(\Omega)$, we shall denote by $(v, Du) \in \mathcal{M}(\Omega)$ the measure defined by

$$\begin{aligned} &(v, Du) := v\,Du \quad \text{in } \ \mathcal{M}(\Omega)\,, \\ &(i.e.\,, \int_\Omega \rho\, d(v, Du) = \int_\Omega \rho\, v\, dDu \text{ for all } \rho \in C_c(\Omega)). \end{aligned}$$

Finally, by $L^1_w(0, T; BV(\Omega))$ (respectively, $L^\infty_w(0, T; BV(\Omega))$) we denote the space of all weakly measurable functions $v : [0, T] \to BV(\Omega)$ such that the mapping $t \mapsto \|v(t)\|_{BV(\Omega)}$ belongs to $L^1([0, T])$ (respectively, $L^\infty([0, T])$). Analogously, the space $L^1_{w*}(0,T;\mathcal{M}(\Omega))$ (respectively, $L^\infty_{w*}(0,T;\mathcal{M}(\Omega))$) consists of all weakly* measurable mappings $\mu:[0,T]\to\mathcal{M}(\Omega)$ such that the function $t\mapsto\|\mu(t)\|_{\mathcal{M}(\Omega)}$ belongs to $L^1([0,T])$ (respectively, $L^\infty([0,T])$).

2.2. Functionals defined on BV

For every $u\in BV(\Omega)$ we consider the following finite Radon measures

$$h(u,Du)=h(u,u_x)\,\mathcal{L}+|D^s u|\,, \tag{2.3}$$

$$f(u,Du)=f(u,u_x)\,\mathcal{L}+|D^s u| \tag{2.4}$$

and, setting $T_{a,b}(s)=\min\{a,\max\{s,b\}\}$ $(a<b)$,

$$h(u,DT_{a,b}(u))=h(u,[T_{a,b}(u)]_x)\,\mathcal{L}+|D^s T_{a,b}(u)|\,; \tag{2.5}$$

here h and f are respectively the functions in (1.5) and in (H_1)–(H_2), and we have denoted by $[T_{a,b}(u)]_x$ the density of the absolutely continuous part of the measure $DT_{a,b}(u)$.

We will also use the following lower semicontinuity result:

Lemma 2.2. *Let $\{u_n\}$ be bounded in $W^{1,1}(\Omega)$ and let $u\in BV(\Omega)$ be such that $u_n\to u$ in $L^1(\Omega)$. Then, for every nonnegative $\rho\in C(\overline{\Omega})$ there holds*

$$\int_\Omega \rho\, df(u,Du)\le \liminf_{n\to\infty}\int_\Omega \rho\, f(u_n,u_{nx})\,\mathrm{d}x\,.$$

Proof. Setting $\tilde f(z,\xi):=f(z,\xi)+D_0$, where D_0 is the constant in (1.3), by the results in [18] (see also [1, Theorem 3.1]), we get

$$\int_\Omega \rho\, d\tilde f(u,Du)\le \liminf_{n\to\infty}\int_\Omega \rho\, \tilde f(u_n,u_{nx})\,\mathrm{d}x\,,$$

where $\tilde f(u,Du)=\tilde f(u,u_x)\,\mathcal{L}+|D^s u|$ in $\mathcal{M}(\Omega)$. Therefore the conclusion follows from last inequality and the definition of $\tilde f(z,\xi)$. □

3. Results

In this section we collect the definitions of sub- and supersolutions to Problem $(P^{u_0}_{\alpha,\beta})$ and the main results of the paper. The proof of these results is the content of the rest of the paper.

3.1. Definitions and comparison

For every $0\le t_0<\tau\le T$ and any $\Omega'\subset\subset\Omega$ we set

$$Q_{t_0,\tau}:=\Omega\times(t_0,\tau),\quad Q'_{t_0,\tau}:=\Omega'\times(t_0,\tau)\,.$$

Let us introduce the notions of *strong sub- and supersolution* to problem $(P^{u_0}_{\alpha,\beta})$.

Definition 3.1. For every $u_0 \in L^1(\Omega)$, by a *strong subsolution (resp. supersolution)* to problem $(P^{u_0}_{\alpha,\beta})$ we mean any function $u \in C([0,T]; L^1(\Omega)) \cap L^1_w(0,T; BV_{\mathrm{loc}}(\Omega))$, $u(0) \leq$ (resp. $\geq$)u_0 *a.e.* in Ω, such that:

(i) $T_{a,b}(u) \in L^1_w(\tau, T; BV(\Omega))$ for all $\tau \in (0,T)$ and for every $a, b \in \mathbb{R}$, $a < b$;

(ii) for every $\tau \in (0,T)$ the distributional derivative u_t belongs to $L^2(Q_{\tau,T})$ and $\mathbf{a}(u, u_x) \in L^2(\tau, T; H^1(\Omega))$;

(iii) for *a.e.* $t \in (0,T)$ there holds

$$u_t(t) \leq (\text{resp.} \geq) \partial_x \big(\mathbf{a}(u(t), u_x(t))\big) + F(t) \quad a.e. \text{ in } \Omega\,,$$
$$\mathbf{a}(u(t), u_x(t))(0) \geq (\text{resp.} \leq) \alpha\,, \quad \mathbf{a}(u(t), u_x(t))(L) \leq (\text{resp.} \geq) \beta\,;$$

(iv) for *a.e.* $t \in (0,T)$ and for every $a, b \in \mathbb{R}$, $a < b$, there holds

$$\big(\mathbf{a}(u(t), u_x(t)), DT_{a,b}(u(t))\big) = h(u(t), DT_{a,b}(u(t))) \quad \text{in } \mathcal{M}(\Omega)\,.$$

By a strong solution to $(P^{u_0}_{\alpha,\beta})$ we mean a strong subsolution which is also a supersolution.

The following feature of strong sub- and supersolutions to problem $(P^{u_0}_{\alpha,\beta})$ is a direct consequence of property (iv) combined with (1.5) and (2.5) (the proof is analogous to that in [22, Proposition 3.2]) and we omit it.

Proposition 3.1. *Let u be a strong sub- or supersolution to $(P^{u_0}_{\alpha,\beta})$. Then, for every interval $\Omega' \subset\subset \Omega$ there holds*

$$D^s_{\pm} u(t) \llcorner \Omega' = D^s_{\pm} u(t) \llcorner \{x \in \Omega' : \boldsymbol{a}(u(t), u_x(t)) = \pm 1\} \quad \textit{for a.e. } t \in (0,T]\,;$$

here $D^s_+ u(t)$ and $D^s_- u(t)$ are respectively the positive and negative parts of $D^s u(t)$.

For every $u_0 \in L^1(\Omega)$, uniqueness of strong solutions to problem $(P^{u_0}_{\alpha,\beta})$ is a direct consequence of the following comparison principle.

Theorem 3.2. *Let $\underline{u}_0, \overline{u}_0 \in L^1(\Omega)$ and let $\underline{u}, \overline{u}$ be respectively a subsolution to problem $(P^{\underline{u}_0}_{\alpha_1,\beta_1})$ and a supersolution to $(P^{\overline{u}_0}_{\alpha_2,\beta_2})$, with $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$. Then, for all $t \in (0,T]$,*

$$\int_\Omega [\underline{u}(t) - \overline{u}(t)]^+ \,\mathrm{d}x \leq \int_\Omega [\underline{u}_0 - \overline{u}_0]^+ \,\mathrm{d}x\,.$$

3.2. Existence and regularity

Existence of strong solutions to problem $(P^{u_0}_{\alpha,\beta})$ for every $u_0 \in L^2(\Omega)$ is ensured by the following theorem.

Theorem 3.3. *For every $u_0 \in L^2(\Omega)$ problem $(P^{u_0}_{\alpha,\beta})$ admits a strong solution*

$$u \in C([0,T]; L^2(\Omega)) \cap L^\infty_w(\tau; T; BV_{\mathrm{loc}}(\Omega)) \quad \textit{for every } \tau \in (0,T)\,.$$

Moreover, in the case that assumption (H_2) holds with $\max\{|\alpha|, |\beta|\} < C_0 \leq 1$, then $u \in L^1_w(0,T; BV(\Omega)) \cap L^\infty_w(\tau, T; BV(\Omega))$ for every $\tau \in (0,T)$.

Strong solutions to problem $(P^{u_0}_{\alpha,\beta})$ exhibit more regularity in the case of BV-initial data.

Theorem 3.4. *For every $u_0 \in BV(\Omega)$, let u be the strong solution to $(P^{u_0}_{\alpha,\beta})$ with initial datum u_0. Then:*

(i) $u \in L^\infty_w(0,T;BV_{\mathrm{loc}}(\Omega))$, $T_{a,b}(u) \in L^1_w(0,T;BV(\Omega))$ *for all $a<b$*, $u_t \in L^2(Q_{0,T})$ *and* $\boldsymbol{a}(u,u_x) \in L^2(0,T;H^1(\Omega))$.

(ii) *If assumption (H_2) holds with* $\max\{|\alpha|,|\beta|\} < C_0 \le 1$, *then* $u \in L^\infty_w(0,T;BV(\Omega))$.

(iii) *For every $\zeta \in C^1([0,T];C^1_c(\Omega))$, $\zeta(\cdot,T)=0$ in Ω, there holds*

$$\int_0^T \left(\int_\Omega \zeta_t(t)\, dDu(t) \right) \mathrm{d}t - \int_{Q_{0,T}} \{[\boldsymbol{a}(u,u_x)]_x\, \zeta_x - F_x\, \zeta\}\, \mathrm{d}x\mathrm{d}t$$
$$= - \int_\Omega \zeta(\cdot,0)\, dDu_0\,. \tag{3.1}$$

Remark 3.1. We explicitly observe that in the case $|\alpha|=1$ or $|\beta|=1$, in general we cannot expect strong solutions to $(P_{\alpha,\beta})$ to belong to the space $L^1_w(0,T;BV(\Omega))$. For instance, it is easy to see that this cannot occur for nonlinearities $\mathbf{a}(z,\xi)$ as in assumption (H_4). Indeed, let $\alpha=1$ (to fix ideas) and, arguing by contradiction, let u be a strong solution to $(P^{u_0}_{\alpha,\beta})$ such that $u(t) \in BV(\Omega)$ (hence $u(t) \in L^\infty(\Omega)$) for $a.e.\ t \in (0,T)$. Without loss of generality, for $a.e.\ t$ as above we may also assume that $\mathbf{a}(u(t),u_x(t)) \in H^1(\Omega)$ (see Definition 3.1–(ii)), whence we get

$$1 - [\mathbf{a}(u(t),u_x(t))](x) = |\,[\mathbf{a}(u(t),u_x(t))](x) - [\mathbf{a}(u(t),u_x(t))](0)\,| \le C(t)\sqrt{|x|}$$

for all $x \in \Omega$, with $C(t) = \|\mathbf{a}(u(t),u_x(t))\|_{H^1(\Omega)}$. On the other hand, by the continuity of the mapping $x \mapsto [\mathbf{a}(u(t),u_x(t))](x)$, and since $[\mathbf{a}(u(t),u_x(t))](0) = 1 > 0$, for all $x \in \Omega$ sufficiently close to $x_0 = 0$ we have $u_x(x,t) > 0$ (observe that, by assumption (H_1), the condition $\mathbf{a}(z,\xi) > 0$ implies $\xi > 0$). Hence, by assumption (H_4)-(i), for $a.e.$ such x there holds

$$\frac{C^+_{3,M}}{(1+u_x(x,t))^\sigma} \le 1 - [\mathbf{a}(u(x,t),u_x(x,t))]\,.$$

Combining the above inequalities gives, in a neighborhood of $x_0 = 0$, the condition $1 + u_x(x,t) \ge \tilde{C}(t)\,|x|^{-1/(2\sigma)}$, for a suitable constant $\tilde{C}(t) > 0$. Since $\sigma \in (0,1/2]$, this implies that $u_x(t) \notin L^1(\Omega)$, a contradiction. Then the conclusion follows.

We finish this section with a stronger regularity result if further assumptions on the flux $\mathbf{a}(z,\xi)$ are assumed. We point out that this result is quite technical, and that Theorem 4.1-(ii) and Proposition 5.5 below are only used for proving it.

Proposition 3.5. *Let $\boldsymbol{a} \in C^1(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$ satisfy, for all $(z,\xi) \in \mathbb{R}^2$,*

$$|\partial_z \boldsymbol{a}(z,\xi)| \le C\sqrt{\partial_\xi \boldsymbol{a}(z,\xi)}\,, \quad \text{for some } C > 0. \tag{3.2}$$

Then, for every strong solution u of problem $(P^{u_0}_{\alpha,\beta})$*, with initial datum* $u_0 \in L^2(\Omega)$*, the distributional derivative* $[\boldsymbol{a}(u,u_x)]_t$ *belongs to* $L^2(Q_{\tau,T})$*. Moreover,* $\boldsymbol{a}(u,u_x) \in L^\infty(\tau,T;H^1(\Omega))$ *for all* $\tau \in (0,T)$*.*

It is worth observing that the conditions $\mathbf{a}(u,u_x) \in L^\infty(\tau,T;H^1(\Omega))$ and $[\mathbf{a}(u,u_x)]_t \in L^2(Q_{\tau,T}))$ for every $\tau \in (0,T)$ imply that $\mathbf{a}(u,u_x) \in C(\overline{\Omega} \times [\tau,T])$ (see [27, Lemma 7.19]).

3.3. Monotonicity properties and regularizing effects for $D^s_\pm u(t)$

Besides (H_1)-(H_2), relying on hypothesis (H_3), we shall prove that for any strong solution u to $(P^{u_0}_{\alpha,\beta})$ with initial data $u_0 \in BV(\Omega)$:

- the singular measures $D^s_\pm(u(t))$ are controlled by $D^s_\pm u_0$ for $a.e.\, t \in (0,T)$;
- the mappings $t \mapsto D^s_\pm(u(t))$ are nonincreasing (in the sense of $\mathcal{M}(\Omega)$).

Theorem 3.6. *Let assumption* (H_3) *be satisfied. For every* $u_0 \in BV(\Omega)$*, let u be the strong solution to* $(P^{u_0}_{\alpha,\beta})$ *with initial datum* u_0*. Then,*

$$D^s_\pm u(t) \le D^s_\pm u_0 \quad \text{in } \mathcal{M}(\Omega) \quad \text{for a.e. } t \in (0,T)\,,$$
$$D^s_\pm u(t_2) \le D^s_\pm u(t_1) \quad \text{in } \mathcal{M}(\Omega) \quad \text{for a.e. } 0 < t_1 < t_2 < T\,.$$

Remark 3.2. For every $\sigma \ge 2$, assumption (H_3) is satisfied with $\sigma_0 = \sigma_1 = \sigma_2 = \sigma$ whenever

$$\mathbf{a}(z,\xi) := \frac{|\xi|^{\sigma-1}\mathrm{sgn}(\xi)}{(\varphi(z) + |\xi|^\sigma)^{1-1/\sigma}}\,, \quad \text{or}$$
$$\mathbf{a}(z,\xi) := \frac{\xi}{(\varphi(z) + |\xi|^\sigma)^{1/\sigma}}$$

with $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\varphi(z) \ge c > 0$ for all $z \in \mathbb{R}$; notice that hypotheses (H_1)–(H_2) hold true as well. We note that (1.9) is easy to prove. We proceed then with (1.10). Since the computations for the two different examples are very similar, we only give the proof for the former case. In this case, (1.10) reads as

$$C^+_{1,M} \le \Psi(z,\xi) \le C^-_{1,M},$$

with $\Psi(z,\xi) := \varphi(z) + \xi^\sigma - \xi^{\sigma-1}(\varphi(z)+\xi^\sigma)^{\frac{1}{\sigma}}$. Therefore, all we need to do is to optimize Ψ in $[-M,M] \times [0,+\infty)$. We compute

$$\Psi_\xi(z,\xi) = \xi^{\sigma-2}(\sigma\xi - (\sigma-1)(\varphi(z)+\xi^\sigma)^{\frac{1}{\sigma}} - \xi^\sigma(\varphi(z)+\xi^\sigma)^{\frac{1}{\sigma}-1}).$$

Then, $\Psi_\xi = 0$ in $(-M,M) \times (0,\infty)$ iff

$$\xi(\varphi(z)+\xi^\sigma)^{\frac{\sigma-1}{\sigma}} = \frac{\sigma-1}{\sigma}\varphi(z) + \xi^\sigma,$$

a contradiction since $\varphi > 0$. Therefore, no interior maxima or minima exist. On the other hand, by L'Hôpital's rule,

$$\lim_{\xi\to+\infty} \Psi(z,\xi) = \lim_{\xi\to+\infty} \frac{(\varphi(z)+\xi^\sigma)^{\frac{\sigma-1}{\sigma}} - \xi^{\sigma-1}}{\frac{1}{(\varphi(z)+\xi^\sigma)^{\frac{1}{\sigma}}}} = \frac{\sigma-1}{\sigma}\varphi(z) < \varphi(z) = \Psi(z,0).$$

Thus, it is enough to consider

$$C_1^+ := \frac{\sigma-1}{\sigma}c < \varphi(z) \le \|\varphi\|_{L^\infty((-M,M))} := C_{1,M}^-, \quad \text{for } |z| \le M.$$

Finally, (1.11) follows from (1.10), since $\mathbf{a}(z,\xi) = -\mathbf{a}(z,-\xi)$. □

Let us show that strong solutions of $(P^{u_0}_{\alpha,\beta})$ exhibit a *regularizing effect* in time—namely, the singular measures $D^s_\pm u(t)$ disappear instantaneously—if the convergence rate of the mapping $\xi \mapsto \mathbf{a}(z,\xi)$ to the saturation values ± 1 (see (1.8)) is not too fast, *i.e.* if assumption (H_4) is satisfied.

Theorem 3.7. *Let u be a strong solution to $(P^{u_0}_{\alpha,\beta})$ with initial data $u_0 \in L^1(\Omega)$.*

(i) *if (H_4)–(i) is satisfied, then for every open interval $\Omega' \subset\subset \Omega$ there holds*

$$D^s_+ u(t) = 0 \quad in \ \mathcal{M}(\Omega') \quad for\ a.e.\ t \in (0,T);$$

(ii) *if (H_4)–(ii) is satisfied, then for every open interval $\Omega' \subset\subset \Omega$ there holds*

$$D^s_- u(t) = 0 \quad in \ \mathcal{M}(\Omega') \quad for\ a.e.\ t \in (0,T).$$

Finally, the following theorem provides sufficient conditions for instantaneous $BV(\Omega)$-$W^{1,1}_{\mathrm{loc}}(\Omega)$ regularizing effects.

Theorem 3.8. *Let assumption (H_3) be satisfied with $\sigma_1, \sigma_2 \in (0,1]$. For any initial data $u_0 \in BV(\Omega)$, let u be the strong solution to $(P^{u_0}_{\alpha,\beta})$. Then, for every open interval $\Omega' \subset\subset \Omega$ there holds*

$$D^s u(t) = 0 \quad in \ \mathcal{M}(\Omega') \quad for\ a.e.\ t \in (0,T).$$

4. Approximating problems

4.1. Well-posedness of the approximating problems

For any $\varepsilon > 0$ and $u_0 \in C^1(\overline{\Omega})$, let us consider the initial-boundary value problem

$$\begin{cases} u_t = v_x + F \ \text{in} \ Q_{0,T}, \\ v(0,t) = \alpha \quad \text{for } t \in [0,T], \\ v(L,t) = \beta \quad \text{for } t \in [0,T], \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where $\alpha,\ \beta \in [-1,1]$ and

$$v := \mathbf{a}(u,u_x) + \sqrt{\varepsilon}u_x + \varepsilon u_{tx} \ \text{ in } \ Q_{0,T}. \tag{4.2}$$

Definition 4.1. For every $u_0 \in C^1(\overline{\Omega})$, by a solution to problem (4.1)–(4.2) we mean any $u \in C^1([0,T]; C^1(\overline{\Omega}))$ such that $v \in C([0,T]; C^2(\overline{\Omega}))$ and the couple (u, v) satisfies (4.1) in the strong sense.

By rephrasing (4.1) as an abstract differential equation in the Banach space $X = C^1(\overline{\Omega})$, we get the following well-posedness result.

Theorem 4.1.

(i) *For every* $u_0 \in C^1(\overline{\Omega})$, $\varepsilon > 0$ *and* $\alpha,\ \beta \in [-1, 1]$ *there exists a unique solution* u *to problem* (4.1)–(4.2). *Moreover,*

$$-\varepsilon v_{xx} + v = \boldsymbol{a}(u, u_x) + \sqrt{\varepsilon} u_x + \varepsilon F_x \quad \text{in } Q_{0,T}\,.$$

(ii) *If, in addition,* $\boldsymbol{a} \in C^1(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$, *we have* $v \in C^1(\overline{Q})$, $v_t \in C([0,T]; C^2(\overline{\Omega}) \cap C_0(\overline{\Omega}))$, *and*

$$-\varepsilon (v_t)_{xx} + v_t = \partial_z \boldsymbol{a}(u, u_x)\, u_t + \partial_\xi \boldsymbol{a}(u, u_x)\, u_{xt} + \sqrt{\varepsilon} u_{xt} + \varepsilon F_{xt} \quad \text{in } Q_{0,T}\,. \tag{4.3}$$

Proof. (i) Let us consider the following abstract ODE in the Banach space $X = C^1(\overline{\Omega})$:

$$\begin{cases} u_t = \mathcal{L}(u, t) \text{ in } (0, T)\,, \\ u(0) = u_0\,, \end{cases} \tag{4.4}$$

where $\mathcal{L} : C^1(\overline{\Omega}) \times [0, T] \to C^1(\overline{\Omega})$ is defined by

$$\mathcal{L}(u, t) = v_x + F(\cdot, t) \qquad (u \in C^1(\overline{\Omega}))\,,$$

$v \in C^2(\overline{\Omega})$ being the unique solution to the problem

$$\begin{cases} -\varepsilon v_{xx} + v = \mathbf{a}(u, u_x) + \sqrt{\varepsilon} u_x + \varepsilon F_x(\cdot, t) \text{ in } \Omega\,, \\ v(0) = \alpha, \quad v(L) = \beta\,. \end{cases} \tag{4.5}$$

Since $\mathbf{a} \in \mathrm{Lip}(\mathbb{R}^2)$, a routine proof shows that $\mathcal{L}$ is continuous in $C^1(\overline{\Omega}) \times [0, T]$ and globally Lipschitz continuous with respect to $u \in C^1(\overline{\Omega})$, uniformly in $t \in [0, T]$. Therefore, for every $u_0 \in C^1(\overline{\Omega})$ there exists a unique solution $u \in C^1([0, T]; C^1(\overline{\Omega}))$ to problem (4.4), satisfying $u_t = v_x + F$ where v is the solution to (4.5). Deriving the equation $u_t = v_x + F$ with respect to x and using (4.5), we get

$$\varepsilon u_{tx} = \varepsilon v_{xx} + \varepsilon F_x = v - \mathbf{a}(u, u_x) - \sqrt{\varepsilon} u_x \;\Rightarrow\; v = \mathbf{a}(u, u_x) + \sqrt{\varepsilon} u_x + \varepsilon u_{tx}\,.$$

This proves that u is a solution to problem (4.1)–(4.2) in the sense of Definition 4.1, whereas the uniqueness part follows by observing that every solution to (4.1)–(4.2) in the sense of Definition 4.1 is also a solution to problem (4.4), which is uniquely solvable.

(ii) In order to show that $v \in C^1(\overline{Q})$, it is enough to prove that for every $(x, t) \in \overline{Q}_T$ there exists the partial derivative $v_t(x, t)$ and $v_t \in C([0, T]; C^2(\overline{\Omega}) \cap C_0(\overline{\Omega}))$. Fix any $t \in (0,T)$ (the cases $t=0$ and $t=T$ can be dealt with similarly). Observe that the function

$$w_h(x) := \frac{v(x,t+h)-v(x,t)}{h} \qquad (0<t+h<T)$$

belongs to $C^2(\overline{\Omega})$ and is the unique solution to the problem

$$\begin{cases} -\varepsilon[w_h]_{xx} + w_h = f_h & \text{in } \Omega \\ w_h = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.6}$$

where, for every $x \in \overline{\Omega}$,

$$f_h(x) = \frac{1}{h}\left[\mathbf{a}(u(x,t+h),u_x(x,t+h)) + \sqrt{\varepsilon}u_x(x,t+h) - \mathbf{a}(u(x,t),u_x(x,t))\right] - \frac{\sqrt{\varepsilon}}{h}u_x(x,t) + \frac{\varepsilon}{h}\left[F_x(x,t+h)-F_x(x,t)\right].$$

Since $u \in C^1([0,T];C^1(\overline{\Omega}))$, $\mathbf{a} \in C^1(\mathbb{R}^2)\cap \mathrm{Lip}(\mathbb{R}^2)$ and $F \in C^2(\overline{Q}_{0,T})$, it can be easily checked that $\{f_h\}$ is uniformly bounded in $C(\overline{\Omega})$, thus $\{w_h\}$ is precompact in $C^1(\overline{\Omega})$. Therefore the conclusion easily follows letting $h \to 0$ in the elliptic problems (4.6) (we omit the standard details). □

Remark 4.1. Let us explicitly notice that under the assumptions of Theorem 4.1-(ii), by (4.2) and the regularity of v_t, it follows that there exists the partial derivative $u_{xtt} \in C(\overline{Q}_{0,T})$ and there holds

$$v_t = \partial_z\mathbf{a}(u,u_x)\,u_t + \partial_\xi\mathbf{a}(u,u_x)\,u_{xt} + \sqrt{\varepsilon}\,u_{xt} + \varepsilon\,u_{xtt}\,. \tag{4.7}$$

For every $a, b \in \mathbb{R}$, $a<b$, set

$$J_{a,b}(z) := \int_0^z T_{a,b}(s)\,\mathrm{d}s, \qquad T_{a,b}(s) = \max\{a,\min\{b,s\}\}\,.$$

We define $r_{\alpha,\beta}$ to be the linear interpolation between α and β in Ω; *i.e.*

$$r_{\alpha,\beta}(x) := \frac{(\beta-\alpha)}{L}x + \alpha \qquad (x \in \overline{\Omega})\,. \tag{4.8}$$

We are ready to prove the following equalities that will be used for obtaining the a priori estimates needed to pass to the limit.

Lemma 4.2. *For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2). Then for every $\zeta \in C^1(\overline{Q}_{0,T})$ and $\tau \in (0,T]$ there holds*

$$\int_{Q_{0,\tau}} h(u,u_x)\,\zeta\,\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{2}\int_\Omega (u_x(\tau))^2\zeta(\tau)\,\mathrm{d}x - \frac{\varepsilon}{2}\int_{Q_{0,\tau}}(u_x)^2\zeta_t\,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon}\int_{Q_{0,\tau}}(u_x)^2\zeta\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} vu\,\zeta_x\,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,\tau}}\frac{u^2}{2}\,\zeta_t\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^\tau \{\beta u(L,t)\zeta(L,t) - \alpha u(0,t)\zeta(0,t)\}\,\mathrm{d}t + \int_\Omega \frac{u_0^2(x)}{2}\,\zeta(x,0)\,\mathrm{d}x \\
&\quad + \frac{\varepsilon}{2}\int_\Omega (u_{0x})^2\zeta(0)\,\mathrm{d}x - \int_\Omega \frac{u^2(x,\tau)}{2}\,\zeta(x,\tau)\,\mathrm{d}x + \int_{Q_{0,\tau}} F u\,\zeta\,\mathrm{d}x\mathrm{d}t\,,
\end{aligned}
$$

where h is the function in (1.5).

Proof. It suffices to multiply the equation $u_t = v_x + F$ by the test function $\eta = u\,\zeta$ and then to integrate by parts. □

Lemma 4.3. *For every* $u_0 \in C^1(\overline{\Omega})$*, let u be the solution to* (4.1)–(4.2)*. Then, for all* $a, b \in \mathbb{R}$*,* $a < b$*, and* $\tau \in (0, T]$ *there hold*

$$
\begin{aligned}
&\int_{Q_{0,\tau}} h(u, [T_{a,b}(u)]_x)\,\zeta\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_{0,\tau}} u_{xt}\,[T_{a,b}(u)]_x\,\zeta\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sqrt{\varepsilon}\int_{Q_{0,\tau}} ([T_{a,b}(u)]_x)^2\zeta\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} v\,T_{a,b}(u)\,\zeta_x\,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,\tau}} J_{a,b}(u)\,\zeta_t\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \left\{\beta T_{a,b}(u(L,t))\zeta(L,t) - \alpha T_{a,b}(u(0,t))\zeta(0,t)\right\}\,\mathrm{d}t \\
&\quad + \int_\Omega J_{a,b}(u_0)\,\zeta(x,0)\,\mathrm{d}x - \int_\Omega J_{a,b}(u(\tau))\,\zeta(x,\tau)\,\mathrm{d}x + \int_{Q_{0,\tau}} F\,T_{a,b}(u)\,\zeta\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{4.9}
$$

for every $\zeta \in C^1(\overline{Q}_{0,T})$*, and*

$$
\begin{aligned}
&\int_{Q_{0,\tau}} t\,h(u, [T_{a,b}(u)]_x)\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_{0,\tau}} t\,u_{xt}\,[T_{a,b}(u)]_x\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sqrt{\varepsilon}\int_{Q_{0,\tau}} t\,([T_{a,b}(u)]_x)^2\mathrm{d}x\mathrm{d}t = \int_0^\tau t\left\{\beta T_{a,b}(u(L,t)) - \alpha T_{a,b}(u(0,t))\right\}\,\mathrm{d}t \\
&\quad - \tau\int_\Omega J_{a,b}(u(\tau))\,\mathrm{d}x + \int_{Q_{0,\tau}} J_{a,b}(u)\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\,F\,T_{a,b}(u)\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
\tag{4.10}
$$

Proof. Equalities (4.9) and (4.10) immediately follow multiplying the equation $u_t = v_x + F$ by the test functions $\eta = T_{a,b}(u)\,\zeta$ and $\eta = t\,T_{a,b}(u)$, respectively and integrating by parts. □

For every solution u of problem (4.1)–(4.2), set

$$
\mathcal{R}(t) := \int_\Omega \left[f(u(t), u_x(t)) - r_{\alpha,\beta}(x)\,u_x(t)\right]\,\mathrm{d}x \qquad (t \in [0, T])\,. \tag{4.11}
$$

Notice that, by the initial condition in (4.1),

$$
\mathcal{R}(0) = \int_\Omega \left[f(u_0, u_{0x}) - r_{\alpha,\beta}(x)u_{0x}\right]\,\mathrm{d}x\,. \tag{4.12}
$$

Here f is the function in (H_1)–(H_2) and $r_{\alpha,\beta}(x)$ is defined in (4.8).

Lemma 4.4. *Let* (H_1)–(H_2) *be satisfied. For every* $u_0 \in C^1(\overline{\Omega})$*, let* u *be the solution to* (4.1)–(4.2)*. Then, for every* $\tau \in (0, T]$ *there holds*

$$\begin{aligned} &\mathcal{R}(\tau) + \int_{Q_{0,\tau}} \left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon \int_{Q_{0,\tau}} (u_{xt})^2 \mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2} \int_\Omega (u_x(\tau))^2 \mathrm{d}x \\ &\quad = \mathcal{R}(0) + \int_{Q_{0,\tau}} \partial_z f(u, u_x)\, u_t \,\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2} \int_\Omega (u_{0x})^2 \mathrm{d}x \\ &\qquad + \int_{Q_{0,\tau}} \left(v - r_{\alpha,\beta}\right) F_x \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\tau\mathcal{R}(\tau) + \int_{Q_{0,\tau}} t\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon \int_{Q_{0,\tau}} t(u_{xt})^2 \mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2} \int_\Omega \tau(u_x(\tau))^2 \mathrm{d}x \\ &= \int_0^\tau \mathcal{R}(t)\mathrm{d}t + \int_{Q_{0,\tau}} t\, \partial_z f(u, u_x)\, u_t \,\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2} \int_{Q_{0,\tau}} (u_x)^2 \mathrm{d}x\mathrm{d}t \\ &\quad + \int_{Q_{0,\tau}} t\left(v - r_{\alpha,\beta}\right) F_x \,\mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{4.14}$$

Proof. Let us firstly prove (4.13). For every $\tau \in (0, T]$ there holds

$$\mathcal{R}(\tau) - \mathcal{R}(0) = \int_{Q_{0,\tau}} \partial_z f(u, u_x)\, u_t \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \left[\partial_\xi f(u, u_x) - r_{\alpha,\beta}(x)\right] u_{xt} \,\mathrm{d}x\mathrm{d}t\,, \tag{4.15}$$

where, by (1.2), (4.2) and the equality $u_{xt} = v_{xx} + F_x$,

$$\begin{aligned} &\int_{Q_{0,\tau}} \left[\partial_\xi f(u, u_x) - r_{\alpha,\beta}(x)\right] u_{xt} \,\mathrm{d}x\mathrm{d}t \\ &\quad = \int_{Q_{0,\tau}} [\mathbf{a}(u, u_x) - v]\, u_{xt} \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \left(v - r_{\alpha,\beta}(x)\right)\left(v_{xx} + F_x\right) \mathrm{d}x\mathrm{d}t \\ &\quad = -\varepsilon \int_{Q_{0,\tau}} (u_{xt})^2 \mathrm{d}x\mathrm{d}t - \sqrt{\varepsilon} \int_{Q_{0,\tau}} u_x\, u_{xt} \,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{Q_{0,\tau}} \left(v - r_{\alpha,\beta}(x)\right)\left(v - r_{\alpha,\beta}(x)\right)_{xx} \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \left(v - r_{\alpha,\beta}(x)\right) F_x \,\mathrm{d}x\mathrm{d}t\,. \end{aligned}$$

Since $v - r_{\alpha,\beta} = 0$ on $\partial\Omega \times (0, T)$, from the above inequality we get

$$\begin{aligned} &\int_{Q_{0,\tau}} \left[\partial_\xi f(u, u_x) - r_{\alpha,\beta}(x)\right] u_{xt} \,\mathrm{d}x\mathrm{d}t \\ &\quad = -\varepsilon \int_{Q_{0,\tau}} (u_{xt})^2 \mathrm{d}x\mathrm{d}t - \sqrt{\varepsilon} \int_{Q_{0,\tau}} u_x\, u_{xt} \,\mathrm{d}x\mathrm{d}t \\ &\qquad - \int_{Q_{0,\tau}} \left[\left(v - r_{\alpha,\beta}(x)\right)_x\right]^2 \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \left(v - r_{\alpha,\beta}(x)\right) F_x \,\mathrm{d}x\mathrm{d}t\,, \end{aligned}$$

which combined with (4.15) gives (4.13).

Let us address (4.14). For every $\tau \in (0, T]$ we have

$$\tau \mathcal{R}(\tau) = \int_0^\tau \mathcal{R}(t)\,\mathrm{d}t + \int_{Q_{0,\tau}} t\,\partial_z f(u, u_x)\, u_t\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\left[\partial_\xi f(u, u_x) - r_{\alpha,\beta}(x)\right] u_{xt}\,\mathrm{d}x\mathrm{d}t\,,$$

and the conclusion follows since, as before,

$$\begin{aligned}
&\int_{Q_{0,\tau}} t\left[\partial_\xi f(u, u_x) - r_{\alpha,\beta}(x)\right] u_{xt}\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_{0,\tau}} t\,[\mathbf{a}(u, u_x) - v]\, u_{xt}\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\big(v - r_{\alpha,\beta}(x)\big)\big(v_{xx} + F_x\big)\,\mathrm{d}x\mathrm{d}t \\
&= -\varepsilon \int_{Q_{0,\tau}} t\big(u_{xt}\big)^2 \mathrm{d}x\mathrm{d}t - \sqrt{\varepsilon}\int_{Q_{0,\tau}} t u_x\, u_{xt}\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{Q_{0,\tau}} t\big(v - r_{\alpha,\beta}(x)\big)\big(v - r_{\alpha,\beta}(x)\big)_{xx}\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\big(v - r_{\alpha,\beta}(x)\big) F_x\,\mathrm{d}x\mathrm{d}t \\
&= -\varepsilon \int_{Q_{0,\tau}} t(u_{xt})^2 \mathrm{d}x\mathrm{d}t - \sqrt{\varepsilon}\int_{Q_{0,\tau}} t u_x\, u_{xt}\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_{0,\tau}} t\left[\big(v - r_{\alpha,\beta}(x)\big)_x\right]^2 \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\big(v - r_{\alpha,\beta}(x)\big) F_x\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}$$

□

We finish this section with some useful inequalities satisfied by the approximating solutions.

For every $g \in C^1(\mathbb{R})$ and $(z, \xi) \in \mathbb{R}^2$, set

$$G_\varepsilon(z, \xi) := \int_0^\xi g(\mathbf{a}_\varepsilon(z, s))\,\mathrm{d}s\,, \quad \text{with } \mathbf{a}_\varepsilon(z, \xi) := \mathbf{a}(z, \xi) + \sqrt{\varepsilon}\,\xi\,. \tag{4.16}$$

Proposition 4.5. *For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2). Then, for every $g \in C^1(\mathbb{R})$, $g' \geq 0$, and $\zeta \in C^1([0, T]; C^1_0(\overline{\Omega}))$, $\zeta \geq 0$, there holds*

$$\begin{aligned}
&\int_{Q_{0,\tau}} G_\varepsilon(u, u_x)\,\zeta_t\,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,\tau}} \Big\{g(v)\,\zeta_x\, v_x + g'(v)\,(v_x)^2\,\zeta - g(v)\,F_x\,\zeta\Big\}\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_\Omega [G_\varepsilon(u, u_x)](x, \tau)\,\zeta(x, \tau)\,\mathrm{d}x - \int_\Omega G_\varepsilon(u_0, u_{0x})\,\zeta(x, 0)\,\mathrm{d}x \\
&\quad - \int_{Q_{0,\tau}} H_{\varepsilon,g}\,\zeta\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.17}$$

for every $\tau \in (0, T]$; here G_ε is the function in (4.16) *and*

$$\begin{aligned}
&H_{\varepsilon,g}(x, t) \\
&:= u_t(x, t)\int_0^{u_x(x,t)} \left[g'\big(\mathbf{a}_\varepsilon(u(x, t), s)\big)\,\partial_z \mathbf{a}(u(x, t), s)\right] \mathrm{d}s \quad ((x, t) \in Q_{0,T}).
\end{aligned} \tag{4.18}$$

Proof. Since $u_{xt} = v_{xx} + F_x$ and $v - \mathbf{a}_\varepsilon(u, u_x) = \varepsilon u_{xt}$ in $Q_{0,T}$, for every $\tau \in (0, T]$ and for every nonnegative $\zeta \in C^1([0, T]; C^1_0(\overline{\Omega}))$, there holds

$$\begin{aligned}
&\int_{Q_{0,\tau}} g\big(\mathbf{a}_\varepsilon(u, u_x)\big)\, u_{xt}\, \zeta \,\mathrm{d}x\mathrm{d}t \\
&\quad = \int_{Q_{0,\tau}} \big[g\big(\mathbf{a}_\varepsilon(u, u_x)\big) - g(v)\big]\, \frac{[v - \mathbf{a}_\varepsilon(u, u_x)]}{\varepsilon}\, \zeta \,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_{Q_{0,\tau}} g(v)\big(v_{xx} + F_x\big)\, \zeta \,\mathrm{d}x\mathrm{d}t \\
&\quad \le - \int_{Q_{0,\tau}} \Big\{ g(v)\, v_x\, \zeta_x + g'(v)\, (v_x)^2\, \zeta - g(v)\, F_x\, \zeta \Big\} \,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

as g is nondecreasing. Moreover, we have

$$\begin{aligned}
[G_\varepsilon(u, u_x)]_t &= g\big(\mathbf{a}_\varepsilon(u, u_x)\big)\, u_{xt} + \int_0^{u_x} \frac{\partial}{\partial t} \big[g\big(\mathbf{a}_\varepsilon(u(x, t), s)\big)\big] \,\mathrm{d}s \\
&= g\big(\mathbf{a}_\varepsilon(u, u_x)\big)\, u_{xt} + H_{\varepsilon,g},
\end{aligned}$$

where $H_{\varepsilon,g}$ is the function in (4.18). Combining the previous inequalities, the conclusion follows. □

5. A priori estimates

5.1. A priori estimates for L^2-initial data

Let us begin by following elementary technical lemma.

Lemma 5.1. *Let $r_{\alpha,\beta}$ be the function in (4.8). Then, for all $z, \xi \in \mathbb{R}$ and $x \in \Omega$ there holds*

$$\min\{\alpha, \beta\} < r_{\alpha,\beta}(x) < \max\{\alpha, \beta\} \quad \textit{for all } x \in \Omega\,, \tag{5.1}$$

$$C_0 \pm r_{\alpha,\beta}(x) > 0 \quad \textit{in } \Omega, \qquad C_0 \pm r_{\alpha,\beta}(x) \ge 0 \quad \textit{on } \partial\Omega\,, \tag{5.2}$$

$$D_0 + f(z, \xi) - r_{\alpha,\beta}(x)\xi \ge \big(C_0 - r_{\alpha,\beta}(x)\big)\xi^+ + \big(C_0 + r_{\alpha,\beta}(x)\big)\xi^- \ge 0\,, \tag{5.3}$$

$$f(z, \xi) - r_{\alpha,\beta}(x)\xi \le h(z, \xi) - r_{\alpha,\beta}(x)\xi + C_1(z^2 + 2)\,, \tag{5.4}$$

where C_0, C_1, D_0 are the constants in assumption (H_2).

Proof. Inequalities (5.1) and (5.2) immediately follow from the very definition of $r_{\alpha,\beta}$ and $((H_2))$, whereas (5.3)–(5.4) are direct consequences of (1.3) and (1.7) and the inequality $|z| + 1 \le z^2 + 2$. □

Remark 5.1. By Lemma 5.1, for every solution u of (4.1)–(4.2) we get, for all $t \in (0, T]$ (see (5.3) and (5.4)),

$$\begin{aligned}&\big(D_0+2C_1\big)|\Omega|+C_1\int_\Omega u^2(t)\mathrm{d}x+\int_\Omega\big\{h(u(t),u_x(t))-r_{\alpha,\beta}(x)\,u_x(t)\big\}\,\mathrm{d}x\\&\quad\ge\int_\Omega\big\{\big(C_0-r_{\alpha,\beta}(x)\big)\,[u_x(t)]^+ +\big(C_0+r_{\alpha,\beta}(x)\big)\,[u_x(t)]^-\big\}\,\mathrm{d}x\ge 0\,,\end{aligned}\tag{5.5}$$

$$\mathcal{R}(t)+D_0|\Omega|\ge\int_\Omega\big\{\big(C_0-r_{\alpha,\beta}(x)\big)[u_x(t)]^+ +\big(C_0+r_{\alpha,\beta}(x)\big)[u_x(t)]^-\big\}\,\mathrm{d}x\ge 0\tag{5.6}$$

and

$$\mathcal{R}(t)\le\int_\Omega\big\{h(u(t),u_x(t))-r_{\alpha,\beta}(x)\,u_x(t)\big\}\,\mathrm{d}x+C_1\int_\Omega u^2(t)\mathrm{d}x+2\,|\Omega|\,;\tag{5.7}$$

here $\mathcal{R}$ is the function in (4.11) and C_0, C_1, D_0 are the constants in assumption (H_2).

Proposition 5.2. *For every $u_0\in C^1(\overline{\Omega})$, $\varepsilon>0$ and α,β as in (1.3), let u be the solution to (4.1)–(4.2). Then, for every $\tau\in(0,T]$ there holds*

$$\int_\Omega u^2(\tau)\,\mathrm{d}x\le D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\},\tag{5.8}$$

$$\int_\Omega u^2(\tau)\,\mathrm{d}x\le\tau D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\}+\int_\Omega\left\{u_0^2+\varepsilon u_{0x}^2\right\}\,\mathrm{d}x\,,\tag{5.9}$$

$$\begin{aligned}&\int_{Q_{0,\tau}}\big\{\big(C_0-r_{\alpha,\beta}(x)\big)[u_x]^+ +\big(C_0+r_{\alpha,\beta}(x)\big)[u_x]^-\big\}\,\mathrm{d}x\mathrm{d}t\\&\quad\le D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\}\end{aligned}\tag{5.10}$$

$$\varepsilon\int_\Omega\big(u_x(\tau)\big)^2\,\mathrm{d}x+\sqrt{\varepsilon}\int_{Q_{0,\tau}}(u_x)^2\mathrm{d}x\mathrm{d}t\le D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\},\tag{5.11}$$

$$\begin{aligned}&\tau\int_\Omega\big\{\big(C_0-r_{\alpha,\beta}(x)\big)[u_x(\tau)]^+ +\big(C_0+r_{\alpha,\beta}(x)\big)[u_x(\tau)]^-\big\}\,\mathrm{d}x\\&\quad\le D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\},\end{aligned}\tag{5.12}$$

$$\begin{aligned}&\varepsilon\int_{Q_{0,\tau}}t\big(u_{xt}\big)^2\mathrm{d}x\mathrm{d}t+\sqrt{\varepsilon}\tau\int_\Omega\big(u_x(\tau)\big)^2\mathrm{d}x\\&\quad+\frac{1}{2}\int_{Q_{0,\tau}}t\left[\big(v-r_{\alpha,\beta}(x)\big)_x\right]^2\mathrm{d}x\mathrm{d}t\le D_{1,T}\left\{1+\int_\Omega u_0^2\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\}\end{aligned}\tag{5.13}$$

$$\int_{Q_{0,\tau}}t(u_t)^2\mathrm{d}x\mathrm{d}t\le D_{1,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x+\varepsilon\int_\Omega(u_{0x})^2\mathrm{d}x\right\}\tag{5.14}$$

for some $D_{1,T}>0$ independent of ε and only depending on α, β, $|\Omega|$, T, F and the constants in (H_1)–(H_2).

Proof. Observe that the following equality holds:

$$\int_0^\tau \{\beta u(L,t)\zeta(L,t) - \alpha u(0,t)\zeta(0,t)\}\,\mathrm{d}t$$
$$= \int_{Q_{0,\tau}} r_{\alpha,\beta}(x)\ (\zeta\, u_x + u\,\zeta_x)\ \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} u\zeta r'_{\alpha,\beta}\mathrm{d}x\mathrm{d}t\,,$$

Therefore, from Lemma 4.2, taking $\zeta = 1$ we get

$$\int_{Q_{0,\tau}} \left\{h(u,u_x) - r_{\alpha,\beta}(x)u_x\right\}\mathrm{d}x\mathrm{d}t + \frac{\epsilon}{2}\int_\Omega \big(u_x(\tau)\big)^2\mathrm{d}x + \sqrt{\varepsilon}\int_{Q_{0,\tau}}(u_x)^2\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega \frac{u^2(\tau)}{2}\mathrm{d}x = \int_\Omega \frac{u_0^2}{2}\mathrm{d}x + \int_{Q_{0,\tau}}\big(F + r'_{\alpha,\beta}(x)\big)u\,\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{2}\int_\Omega (u_{0x})^2\mathrm{d}x\,, \tag{5.15}$$

whence (see (5.5))

$$\int_{Q_{0,\tau}} \overbrace{\left\{\big(C_0 - r_{\alpha,\beta}(x)\big)[u_x]^+ + \big(C_0 + r_{\alpha,\beta}(x)\big)[u_x]^-\right\}}^{\geq 0}\mathrm{d}x\mathrm{d}t$$
$$+\frac{\epsilon}{2}\int_\Omega (u_x(\tau))^2\mathrm{d}x + \sqrt{\varepsilon}\int_{Q_{0,\tau}}(u_x)^2\mathrm{d}x\mathrm{d}t + \int_\Omega\frac{u^2(\tau)}{2}\mathrm{d}x \le \int_\Omega \frac{u_0^2}{2}\mathrm{d}x$$
$$+\int_{Q_{0,\tau}}\big(F + r'_{\alpha,\beta}(x)\big)u\,\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{2}\int_\Omega (u_{0x})^2\mathrm{d}x + \big(D_0 + 2C_1\big)|\Omega|\tau$$
$$+C_1\int_{Q_{0,\tau}} u^2\mathrm{d}x\mathrm{d}t \le A_1\tau + A_2\int_{Q_{0,\tau}} u^2\mathrm{d}x\mathrm{d}t + \int_\Omega\frac{u_0^2}{2}\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega (u_{0x})^2\mathrm{d}x\,, \tag{5.16}$$

by Young's inequality. Here

$$A_1 = \big(D_0 + 2C_1\big)\,|\Omega| + \frac{\|F + r'_{\alpha,\beta}\|^2_{L^\infty(0,T;L^2(\Omega))}}{2}\,, \qquad A_2 = \frac{1}{2} + C_1\,.$$

Hence, (5.8) follows from (5.16) and Gronwall's Lemma, whereas (5.9), (5.10) and (5.11) are a direct consequence of (5.16) and (5.8). Moreover, by (5.15) and (5.8) we also get

$$\int_{Q_{0,\tau}}\left\{h(u,u_x) - r_{\alpha,\beta}(x)\,u_x\right\}\mathrm{d}x\mathrm{d}t \le \tilde{D}_{1,T}\left\{1 + \int_\Omega u_0^2\,\mathrm{d}x + \varepsilon\int_\Omega (u_{0x})^2\mathrm{d}x\right\} \tag{5.17}$$

for some $\tilde{D}_{1,T} > 0$ independent of ε and only depending on $\alpha,\ \beta,\ |\Omega|,\ T$, F and the constants in (H_1)–(H_2). Let us now address the remaining estimates (5.12)–(5.14). To this aim, let us first notice that by the equality $u_t = v_x + F$ there holds

$$u_t = (v - r_{\alpha,\beta})_x + r'_{\alpha,\beta} + F\,, \tag{5.18}$$

whereas by (4.14) and (5.6) it follows that, for all $\tau \in (0, T]$,

$$
\begin{aligned}
&-\tau D_0\,|\Omega| + \tau \int_\Omega \left\{(C_0 - r_{\alpha,\beta}(x))[u_x(\tau)]^+ + (C_0 + r_{\alpha,\beta}(x))[u_x(\tau)]^-\right\} \mathrm{d}x \\
&\quad + \int_{Q_{0,\tau}} t\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon \int_{Q_{0,\tau}} t(u_{xt})^2 \mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\tau \int_\Omega (u_x(\tau))^2 \mathrm{d}x \\
&\leq \int_0^\tau \mathcal{R}(t)\,\mathrm{d}t + \int_{Q_{0,\tau}} t\,\partial_z f(u, u_x)\,u_t\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\sqrt{\varepsilon}}{2} \int_{Q_{0,\tau}} (u_x)^2 \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t(v - r_{\alpha,\beta})\,F_x\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
\tag{5.19}
$$

Concerning the right-hand side in the above inequality, we have

$$
\int_0^\tau \mathcal{R}(t)\,\mathrm{d}t \leq \tilde{D}_{1,T}\left\{1 + \int_\Omega u_0^2\,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right\} + C_1 \int_{Q_{0,\tau}} u^2 \mathrm{d}x\mathrm{d}t + 2|\Omega|\tau
\tag{5.20}
$$

(see (5.7) and (5.17)) and, by (5.18)

$$
\begin{aligned}
&\int_{Q_{0,\tau}} t\,\partial_z f(u, u_x)\,u_t\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t(v - r_{\alpha,\beta})\,F_x\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_{0,\tau}} t\partial_z f(u, u_x)\,(r'_{\alpha,\beta} + F)\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t\left(\partial_z f(u, u_x) - F\right)(v - r_{\alpha,\beta})_x\,\mathrm{d}x\mathrm{d}t \\
&\leq \frac{\tau^2}{2}\gamma \|r'_{\alpha,\beta} + F\|_{L^\infty(0,T;L^1(\Omega))} + \int_{Q_{0,\tau}} t\,\frac{\left(\partial_z f(u, u_x) - F\right)^2}{2} \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{Q_{0,\tau}} t\,\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t \leq A_3 + \frac{1}{2}\int_{Q_{0,\tau}} t\,\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{5.21}
$$

(here we have also used (1.2)), for some $A_3 > 0$ independent of ε and only depending on $\alpha,\ \beta,\ |\Omega|,\ T,\ F$ and the constants in (H_1)–(H_2). Combining (5.19)–(5.21) gives

$$
\begin{aligned}
&\tau \int_\Omega \left\{(C_0 - r_{\alpha,\beta}(x))[u_x(\tau)]^+ + (C_0 + r_{\alpha,\beta}(x))[u_x(\tau)]^-\right\} \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{Q_{0,\tau}} t\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon \int_{Q_{0,\tau}} t\left(u_{xt}\right)^2 \mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\tau \int_\Omega (u_x(\tau))^2 \mathrm{d}x \\
&\leq \tilde{D}_{1,T}\left\{1 + \int_\Omega u_0^2 \mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right\} \\
&\quad + \frac{\sqrt{\varepsilon}}{2} \int_{Q_{0,\tau}} (u_x)^2 \mathrm{d}x\mathrm{d}t + C_1 \int_{Q_{0,\tau}} u^2 \mathrm{d}x\mathrm{d}t + A_4,
\end{aligned}
\tag{5.22}
$$

for some $A_4 > 0$ independent of ε. Therefore (5.12) and (5.13) follow from (5.22), (5.8) and (5.11). The proof of (5.14) relies on (5.18) and the estimate of the term $\int_{Q_{0,\tau}} t\left[(v - r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t$ in (5.13). □

Proposition 5.3. *For every $u_0 \in C^1(\overline{\Omega})$, $\varepsilon > 0$ and α, β as in* (1.3)*, let u be the solution to* (4.1)–(4.2).

(i) *For every interval $\Omega' \subset\subset \Omega$ there exists $D_{\Omega',T} > 0$, independent of ε, such that*

$$\int_{Q'_{0,T}} |u_x| \,\mathrm{d}x\mathrm{d}t \leq D_{\Omega',T} \left(1 + \int_\Omega u_0^2 \,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right), \tag{5.23}$$

$$\int_{\Omega'} |u_x(\tau)| \,\mathrm{d}x \leq \frac{D_{\Omega',T}}{\tau} \left(1 + \int_\Omega u_0^2 \,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right) \quad \textit{for every } \tau \in (0,T]\,. \tag{5.24}$$

(ii) *If (H_2) holds with $\max\{|\alpha|, |\beta|\} < C_0 \leq 1$, there exists $D_{\Omega,T} > 0$ independent of ε such that*

$$\int_{Q_{0,T}} |u_x| \,\mathrm{d}x\mathrm{d}t \leq D_{\Omega,T} \left(1 + \int_\Omega u_0^2 \,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right), \tag{5.25}$$

$$\int_{\Omega} |u_x(\tau)| \,\mathrm{d}x \leq \frac{D_{\Omega,T}}{\tau} \left(1 + \int_\Omega u_0^2 \,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right) \quad \textit{for every } \tau \in (0,T]\,. \tag{5.26}$$

Proof.

(i) Estimates (5.23) and (5.24) follow from (5.10) and (5.12), respectively, combined with the definition of $r_{\alpha,\beta}$ and $(H2)$.

(ii) The proof of (5.25) and (5.26) relies on (5.10) and (5.12), respectively, combined with the requirement $C_0 > \max\{|\alpha|, |\beta|\}$, which plainly gives $C_0 \pm r_{\alpha,\beta} \geq c > 0$ in $\overline{\Omega}$.

□

Proposition 5.4. *For every $u_0 \in C^1(\overline{\Omega})$, $\varepsilon > 0$ and α, β as in (1.3), let u be the solution to (4.1)–(4.2). Then, for all $a, b \in \mathbb{R}$ with $a < b$, there holds*

$$\int_{Q_T} t \left|[T_{a,b}(u)]_x\right| \,\mathrm{d}x\mathrm{d}t \leq D_{a,b,T} \left(1 + \int_\Omega u_0^2 \,\mathrm{d}x + \varepsilon \int_\Omega (u_{0x})^2 \mathrm{d}x\right), \tag{5.27}$$

for some $D_{a,b,T} > 0$ independent of ε and only depending on $a,\ b,\ \alpha,\ \beta,\ |\Omega|,\ T,\ F$ and the constants in assumptions (H_1)–(H_2).

Proof. By (4.10) we get

$$\begin{aligned}
&\int_{Q_{0,T}} t\, h(u, [T_{a,b}(u)]_x) \,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon} \int_{Q_{0,T}} t \left([T_{a,b}(u)]_x\right)^2 \mathrm{d}x\mathrm{d}t \\
&\leq \varepsilon \int_{Q_{0,T}} t\, |\, u_{xt}\, [T_{a,b}(u)]_x \,|\, \mathrm{d}x\mathrm{d}t + \int_0^T t\, |\, \beta T_{a,b}(u(L,t)) - \alpha T_{a,b}(u(0,t)) \,|\, \mathrm{d}t \\
&\quad + T \int_\Omega |J_{a,b}(u(T))| \,\mathrm{d}x + \int_{Q_{0,T}} |J_{a,b}(u)| \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} t\, |F|\, |T_{a,b}(u)| \,\mathrm{d}x\mathrm{d}t\,.
\end{aligned} \tag{5.28}$$

Concerning the right-hand side of (5.28) we observe that

$$\int_0^T t\,|\,\beta T_{a,b}(u(L,t)) - \alpha T_{a,b}(u(0,t))\,|\,\mathrm{d}t + \int_{Q_{0,T}} t\,|F|\,|T_{a,b}(u)|\,\mathrm{d}x\mathrm{d}t \le A_1\,, \tag{5.29}$$

$$\begin{aligned} &T\int_\Omega |J_{a,b}(u(T))|\,\mathrm{d}x + \int_{Q_{0,T}} |J_{a,b}(u)|,\mathrm{d}x\mathrm{d}t \\ &\quad\le A_1\left(1+\int_\Omega u^2(T)\,\mathrm{d}x + \int_{Q_{0,T}} u^2\mathrm{d}x\mathrm{d}t\right), \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\varepsilon\int_{Q_{0,T}} t\,|\,u_{xt}\,[T_{a,b}(u)]_x\,|\,\mathrm{d}x\mathrm{d}t \\ &\quad\le \frac{\varepsilon^{3/2}}{2}\int_{Q_{0,T}} t\,(u_{xt})^2\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_{Q_{0,T}} t\,\big([T_{a,b}(u)]_x\big)^2\mathrm{d}x\mathrm{d}t\,, \end{aligned} \tag{5.31}$$

for some $A_1 > 0$ independent of ε and only depending on $a,\ b,\ \alpha,\ \beta,\ |\Omega|,\ T,\ F$ and the constants in assumptions (H_1)–(H_2). Using (5.29)–(5.31), inequality (5.28) reads as

$$\begin{aligned} &\int_{Q_{0,T}} t\,h(u,[T_{a,b}(u)]_x)\,\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_{Q_{0,T}} t\,\big([T_{a,b}(u)]_x\big)^2\,\mathrm{d}x\mathrm{d}t \\ &\quad\le \frac{\varepsilon^{3/2}}{2}\int_{Q_{0,T}} t\,\big(u_{xt}\big)^2\mathrm{d}x\mathrm{d}t + A_1\left(2+\int_\Omega u^2(T)\,\mathrm{d}x + \int_{Q_{0,T}} u^2\mathrm{d}x\mathrm{d}t\right) \\ &\quad\le A_2\left(1+\int_\Omega u_0^2\,\mathrm{d}x + \varepsilon\int_\Omega (u_{0x})^2\mathrm{d}x\right) \end{aligned} \tag{5.32}$$

(here we have used (5.8) and (5.13)). Then we obtain (5.27) from (5.32) since, by (1.7), $h(u,[T_{a,b}(u)]_x) \ge C_0|[T_{a,b}(u)]_x| - D_0 - C_1(|u|+1)$ (see also (5.8)). □

Proposition 5.5. *Let $\boldsymbol{a} \in C^1(\mathbb{R}^2;\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})$ satisfy (3.2). For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2). Then, for every $\tau \in (0,T]$ there holds*

$$\begin{aligned} &\frac{1}{2}\int_{Q_{0,\tau}} t^2(u_{xt})^2\partial_\xi \boldsymbol{a}(u,u_x)\,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon}\int_{Q_{0,\tau}} t^2(u_{xt})^2\mathrm{d}x\mathrm{d}t + \frac{\varepsilon\tau^2}{4}\int_\Omega (u_{xt})^2(\tau)\,\mathrm{d}x \\ &\quad+\frac{\tau^2}{4}\int_\Omega \left[(v-r_{\alpha,\beta})_x\right]^2(\tau)\,\mathrm{d}x \le D_{2,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x + \varepsilon\int_\Omega (u_{0x})^2\mathrm{d}x\right\}, \end{aligned} \tag{5.33}$$

$$\int_{Q_{0,T}} t^2\left[\partial_t\big(\boldsymbol{a}(u,u_x)\big)\right]^2\mathrm{d}x\mathrm{d}t \le D_{2,T}\left\{1+\int_\Omega u_0^2\,\mathrm{d}x + \varepsilon\int_\Omega (u_{0x})^2\mathrm{d}x\right\}, \tag{5.34}$$

for some $D_{2,T} > 0$ independent of ε only depending on $\alpha,\beta,\ |\Omega|,\ T,\ F$ and the constants in assumptions (H_1)–(H_2).

Proof. In view of Theorem 4.1-(ii) we can multiply the equation $u_{xt} = v_{xx} + F_x = (v - r_{\alpha,\beta})_{xx} + F_x$ by the test function $\eta = t^2 v_t = t^2(v - r_{\alpha,\beta})_t$. Then, for every $\tau \in (0, T]$ we get

$$\begin{aligned}
&\int_{Q_{0,\tau}} t^2 u_{xt}\, v_t \,\mathrm{d}x\mathrm{d}t\\
&= \int_{Q_{0,\tau}} t^2 (v - r_{\alpha,\beta})_t\, (v - r_{\alpha,\beta})_{xx} \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t^2 (v - r_{\alpha,\beta})_t\, F_x \,\mathrm{d}x\mathrm{d}t\\
&= - \int_{Q_{0,\tau}} t^2 (v - r_{\alpha,\beta})_{tx}\, (v - r_{\alpha,\beta})_x \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \partial_t \big(t^2 (v - r_{\alpha,\beta})\, F_x\big) \,\mathrm{d}x\mathrm{d}t\\
&\quad - \int_{Q_{0,\tau}} 2t\, (v - r_{\alpha,\beta})\, F_x \,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,\tau}} t^2\, (v - r_{\alpha,\beta})\, F_{xt} \,\mathrm{d}x\mathrm{d}t\\
&= - \frac{\tau^2}{2} \int_\Omega \big[(v - r_{\alpha,\beta})_x\big]^2 (\tau) \,\mathrm{d}x\\
&\quad + \int_{Q_{0,\tau}} t\, \big[(v - r_{\alpha,\beta})_x\big]^2 \,\mathrm{d}x\mathrm{d}t - \tau^2 \int_\Omega (v(\tau) - r_{\alpha,\beta})_x\, F(x,\tau) \,\mathrm{d}x\\
&\quad + 2 \int_{Q_{0,\tau}} t\, (v - r_{\alpha,\beta})_x\, F \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t^2\, (v - r_{\alpha,\beta})_x\, F_t \,\mathrm{d}x\mathrm{d}t\,,
\end{aligned} \tag{5.35}$$

(here we have used that $v - r_{\alpha,\beta} = 0$ and $v_t = 0$ on $\partial\Omega \times (0, \tau)$). Concerning the left-hand side of (5.35), we have (see (4.7))

$$\begin{aligned}
&\int_{Q_{0,\tau}} t^2 u_{xt}\, v_t \,\mathrm{d}x\mathrm{d}t\\
&= \int_{Q_{0,\tau}} t^2 u_{xt}\, u_t\, \partial_z \mathbf{a}(u, u_x) \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t^2 (u_{xt})^2 \partial_\xi \mathbf{a}(u, u_x) \,\mathrm{d}x\mathrm{d}t\\
&\quad + \sqrt{\varepsilon} \int_{Q_{0,\tau}} t^2 (u_{xt})^2 \mathrm{d}x\mathrm{d}t + \frac{\varepsilon\tau^2}{2} \int_\Omega (u_{xt})^2 (\tau) \,\mathrm{d}x - \varepsilon \int_{Q_{0,\tau}} t\, (u_{xt})^2 \mathrm{d}x\mathrm{d}t\,.
\end{aligned} \tag{5.36}$$

Combining (5.35) and (5.36), since

$$\begin{aligned}
&\tau^2 \left| \int_\Omega (v(\tau) - r_{\alpha,\beta})_x\, F(x,\tau) \,\mathrm{d}x \right|\\
&\quad \le \frac{\tau^2}{4} \int_\Omega [(v(\tau) - r_{\alpha,\beta})_x]^2 \,\mathrm{d}x + \tau^2 \int_\Omega F^2(x,\tau) \,\mathrm{d}x\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{Q_{0,\tau}} t^2 (u_{xt})^2 \partial_\xi \mathbf{a}(u, u_x) \,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon} \int_{Q_{0,\tau}} t^2 (u_{xt})^2 \mathrm{d}x\mathrm{d}t + \frac{\varepsilon\tau^2}{2} \int_\Omega (u_{xt})^2 (\tau) \,\mathrm{d}x\\
&\quad + \frac{\tau^2}{4} \int_\Omega \big[(v - r_{\alpha,\beta})_x\big]^2 (\tau) \,\mathrm{d}x \le \int_{Q_{0,\tau}} t^2 |u_{xt}|\, |u_t|\, |\partial_z \mathbf{a}(u, u_x)| \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_{0,\tau}} t\,\left[(v-r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + 2\int_{Q_{0,\tau}} t\,(v-r_{\alpha,\beta})_x\, F\,\mathrm{d}x\mathrm{d}t \\
&+\varepsilon\int_{Q_{0,\tau}} t\,(u_{xt})^2\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} t^2\,(v-r_{\alpha,\beta})_x\, F_t\,\mathrm{d}x\mathrm{d}t + \tau^2\int_{\Omega} F^2(x,\tau)\,\mathrm{d}x \\
&\le A_1\left\{1+\int_\Omega u_0^2\,\mathrm{d}x + \varepsilon\int_\Omega (u_{0x})^2\mathrm{d}x\right\} + \int_{Q_{0,\tau}} t^2|u_{xt}|\,|u_t|\,|\partial_z \mathbf{a}(u,u_x)|\,\mathrm{d}x\mathrm{d}t\,,
\end{aligned} \tag{5.37}
$$

the latter inequality in the above estimate following by (5.13); here $A_1 > 0$ is a suitable constant independent of ε, only depending on α, β, $|\Omega|$, T, F and the constants in assumptions (H_1)–(H_2). By (3.2) we get

$$
\begin{aligned}
&\int_{Q_{0,\tau}} t^2|u_{xt}|\,|u_t|\,|\partial_z \mathbf{a}(u,u_x)|\,\mathrm{d}x\mathrm{d}t \\
&\quad\le C_a\int_{Q_{0,\tau}} t^2|u_{xt}|\,|u_t|\,\sqrt{\partial_\xi \mathbf{a}(u,u_x)}\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \frac{1}{2}\int_{Q_{0,\tau}} t^2(u_{xt})^2\partial_\xi \mathbf{a}(u,u_x)\,\mathrm{d}x\mathrm{d}t + \frac{C_a^2}{2}\int_{Q_{0,\tau}} t^2(u_t)^2\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and (5.33) follows immediately from the above inequality, (5.37) and (5.14).

Finally, (5.34) is a direct consequence of (5.14), (5.33) and the inequality

$$
\begin{aligned}
&t^2\left[\partial_t\big(\mathbf{a}(u,u_x)\big)\right]^2 \\
&\le 2t^2\big(\partial_z \mathbf{a}(u,u_x)\big)^2(u_t)^2 + 2t^2\big(\partial_\xi \mathbf{a}(u,u_x)\big)^2(u_{xt})^2 \\
&\le 2t^2\left[\sup_{z,\xi\in\mathbb{R}^2}|\partial_z \mathbf{a}(z,\xi)|\right]^2(u_t)^2 + 2t^2\left[\sup_{(z,\xi)\in\mathbb{R}^2}\partial_\xi \mathbf{a}(z,\xi)\right]\partial_\xi \mathbf{a}(u,u_x)\,(u_{xt})^2
\end{aligned}
$$

(recall that $\mathbf{a}\in \mathrm{Lip}(\mathbb{R}^2)$). □

5.2. A priori estimates for BV-initial data

Proposition 5.6. *For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2). Then, for every $\tau\in(0,T]$ there holds*

$$
\begin{aligned}
&\int_\Omega \left\{\big(C_0 - r_{\alpha,\beta}(x)\big)[u_x(\tau)]^+ + \big(C_0 + r_{\alpha,\beta}(x)\big)[u_x(\tau)]^-\right\}\,\mathrm{d}x \\
&\quad\le \bar{D}_{1,T}\left(1+\|u_0\|_{BV(\Omega)} + \sqrt{\varepsilon}\int_\Omega (u_{0x})^2\mathrm{d}x\right),
\end{aligned} \tag{5.38}
$$

$$
\begin{aligned}
&\varepsilon\int_{Q_{0,\tau}} \big(u_{xt}\big)^2\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_\Omega \big(u_x(\tau)\big)^2\mathrm{d}x + \frac{1}{2}\int_{Q_{0,\tau}}\left[\big(v-r_{\alpha,\beta}(x)\big)_x\right]^2\mathrm{d}x\mathrm{d}t \\
&\quad\le \bar{D}_{1,T}\left(1+\|u_0\|_{BV(\Omega)} + \sqrt{\varepsilon}\int_\Omega (u_{0x})^2\mathrm{d}x\right),
\end{aligned} \tag{5.39}
$$

$$\int_{Q_{0,\tau}} \big(u_t\big)^2 \mathrm{d}x\mathrm{d}t \le \bar{D}_{1,T}\left(1+\|u_0\|_{BV(\Omega)}+\sqrt{\varepsilon}\int_\Omega (u_{0x})^2\mathrm{d}x\right), \tag{5.40}$$

for some $\bar{D}_{1,T}>0$ independent of ε and only depending on $\alpha,\ \beta,\ |\Omega|,\ T,\ F$ and the constants in assumptions (H_1)–(H_2).

Proof. Combining (4.13) and (5.6) gives, for all $\tau \in (0,T]$,

$$\begin{aligned}
&-D_0\,|\Omega| + \int_\Omega \left\{(C_0 - r_{\alpha,\beta}(x))[u_x(\tau)]^+ + (C_0 + r_{\alpha,\beta}(x))[u_x(\tau)]^-\right\}\mathrm{d}x \\
&\quad + \int_{Q_{0,\tau}} \left[(v-r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_{0,\tau}} \big(u_{xt}\big)^2\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_\Omega \big(u_x(\tau)\big)^2\mathrm{d}x \\
&\le \mathcal{R}(0) + \int_{Q_{0,\tau}} \partial_z f(u,u_x)\,u_t\,\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_\Omega (u_{0x})^2\mathrm{d}x \\
&\quad + \int_{Q_{0,\tau}} (v-r_{\alpha,\beta})\,F_x\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned} \tag{5.41}$$

Concerning the right-hand side in the above inequality, there exists $A_1>0$ independent of ε, such that

$$\mathcal{R}(0) = \int_\Omega \left[f(u_0,u_{0x}) - r_{\alpha,\beta}(x)u_{0x}\right]\mathrm{d}x \le A_1\big(1+\|u_0\|_{BV(\Omega)}\big) \tag{5.42}$$

(see (1.3) and (4.12)) and, by (5.18)

$$\begin{aligned}
&\int_{Q_{0,\tau}} \partial_z f(u,u_x)\,u_t\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} (v-r_{\alpha,\beta})\,F_x\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_{0,\tau}} \partial_z f(u,u_x)\,(r'_{\alpha,\beta}+F)\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} \big(\partial_z f(u,u_x)-F\big)(v-r_{\alpha,\beta})_x\,\mathrm{d}x\mathrm{d}t \\
&\le \tau\gamma\|r'_{\alpha,\beta}+F\|_{L^\infty(0,T;L^1(\Omega))} + \int_{Q_{0,\tau}} \frac{\big(\partial_z f(u,u_x)-F\big)^2}{2}\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{Q_{0,\tau}} \left[(v-r_{\alpha,\beta})_x\right]^2\mathrm{d}x\mathrm{d}t \le A_2 + \frac{1}{2}\int_{Q_{0,\tau}} \left[(v-r_{\alpha,\beta})_x\right]^2\mathrm{d}x\mathrm{d}t\,,
\end{aligned} \tag{5.43}$$

(see also (1.2)), for some $A_2>0$ independent of ε. Combining (5.42)–(5.43), inequality (5.41) can be rephrased as

$$\begin{aligned}
&\int_\Omega \left\{(C_0 - r_{\alpha,\beta}(x))[u_x(\tau)]^+ + (C_0 + r_{\alpha,\beta}(x))[u_x(\tau)]^-\right\}\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{Q_{0,\tau}} \left[(v-r_{\alpha,\beta})_x\right]^2 \mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_{0,\tau}} \big(u_{xt}\big)^2\mathrm{d}x\mathrm{d}t + \frac{\sqrt{\varepsilon}}{2}\int_\Omega \big(u_x(\tau)\big)^2\mathrm{d}x \\
&\le A_1\big(1+\|u_0\|_{BV(\Omega)}\big) + A_2 + \frac{\sqrt{\varepsilon}}{2}\int_\Omega (u_{0x})^2\mathrm{d}x\,,
\end{aligned}$$

whence (5.38) and (5.39) immediately follow. Finally, (5.40) is a direct consequence of (5.18) and the estimate of the term $\int_{Q_{0,\tau}}\left[(v-r_{\alpha,\beta})_x\right]^2\mathrm{d}x\mathrm{d}t$ in (5.39). □

Proposition 5.7. *For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2).*

(i) *For every interval $\Omega' \subset\subset \Omega$ there exists $\bar{D}_{\Omega',T} > 0$ independent of ε such that*

$$\int_{\Omega'} |u_x(\tau)|\,\mathrm{d}x \leq \bar{D}_{\Omega',T}\left(1 + \|u_0\|_{BV(\Omega)} + \sqrt{\varepsilon}\int_{\Omega}(u_{0x})^2\mathrm{d}x\right), \quad \textit{for every } \tau \in (0,T]. \tag{5.44}$$

(ii) *If assumption (H_2) holds with $\max\{|\alpha|, |\beta|\} < C_0 \leq 1$, then there exists $\bar{D}_{\Omega,T} > 0$ independent of ε such that*

$$\int_{\Omega} |u_x(\tau)|\,\mathrm{d}x \leq \bar{D}_{\Omega,T}\left(1 + \|u_0\|_{BV(\Omega)} + \sqrt{\varepsilon}\int_{\Omega}(u_{0x})^2\mathrm{d}x\right), \quad \textit{for every } \tau \in (0,T]. \tag{5.45}$$

Proof. The proof is analogous to that of Proposition 5.3, by using (5.38) instead of (5.12) (we omit the details). □

Proposition 5.8. *For every $u_0 \in C^1(\overline{\Omega})$, let u be the solution to (4.1)–(4.2). Then, for all $a, b \in \mathbb{R}$ with $a < b$, there holds*

$$\int_{Q_{0,T}} \left|[T_{a,b}(u)]_x\right|\,\mathrm{d}x\mathrm{d}t \leq \bar{D}_{a,b,T}\left(1 + \|u_0\|_{BV(\Omega)} + \sqrt{\varepsilon}\int_{\Omega}(u_{0x})^2\mathrm{d}x\right). \tag{5.46}$$

for some $\bar{D}_{a,b,T} > 0$ independent of ε.

Proof. The proof is completely analogous to that of Proposition 5.4, relying on (4.9) instead of (4.10) (we omit the details). □

6. Proof of comparison and existence

6.1. Proof of comparison

Let us prove the comparison principle for sub- and supersolutions to $(P^{u_0}_{\alpha,\beta})$.

Proof of Theorem 3.2. Let $\underline{u}$ be a strong subsolution to problem $(P^{\underline{u}_0}_{\alpha_1,\beta_1})$, and let $\overline{u}$ be a strong supersolution to problem $(P^{\overline{u}_0}_{\alpha_2,\beta_2})$, with data $\underline{u}_0 \in L^1(\Omega)$ and $\overline{u}_0 \in L^1(\Omega)$. For simplicity set $\underline{\mathbf{z}} := \mathbf{a}(\underline{u}, \underline{u}_x)$ and $\overline{\mathbf{z}} := \mathbf{a}(\overline{u}, \overline{u}_x)$.

For every $j \in \mathbb{N}$ large enough, let $\rho_j \in W^{1,\infty}(\Omega) \cap C_c(\Omega)$ be defined by setting

$$\rho_j(x) = j(x - 1/j)\chi_{\{\frac{1}{j}<x<\frac{2}{j}\}} + \chi_{\{\frac{2}{j}\leq x\leq L-\frac{2}{j}\}} - j(x - L + 1/j)\chi_{\{L-\frac{2}{j}<x<L-\frac{1}{j}\}}.$$

Then Definition 3.1-(iii) and the condition $\underline{u}, \overline{u} \in L^1_w(0,T; BV_{\mathrm{loc}}(\Omega))$ ensure that, for every $K > 0$ and $0 < t_0 < \tau < T$ there holds

$$\begin{aligned}
&\int_{Q_{t_0,\tau}} (\underline{u}-\overline{u})_t\, T_{0,K}(\underline{u}-\overline{u})\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le -\int_{t_0}^{\tau}\left(\int_{\Omega}\rho_j(x)\,\big(\underline{\mathbf{z}}(t)-\overline{\mathbf{z}}(t)\big)\,dDT_{0,K}(\underline{u}(t)-\overline{u}(t))\right)\mathrm{d}t \\
&\qquad -\int_{Q_{t_0,\tau}}\rho_j'(x)(\underline{\mathbf{z}}-\overline{\mathbf{z}})\,T_{0,K}(\underline{u}-\overline{u})\,\mathrm{d}x\mathrm{d}t\,. \qquad (6.1)
\end{aligned}$$

On the other hand, since $\alpha_1 \ge \alpha_2$ and $\beta_1 \le \beta_2$, we have

$$\begin{aligned}
&-\int_{Q_{t_0,\tau}}\rho_j'(x)(\underline{\mathbf{z}}-\overline{\mathbf{z}})\,T_{0,K}(\underline{u}-\overline{u})\,\mathrm{d}x\mathrm{d}t \\
&= j\int_{t_0}^{\tau}\int_{L-2/j}^{L-1/j}\big((\underline{\mathbf{z}}-\beta_1)+(\beta_1-\beta_2)+(\beta_2-\overline{\mathbf{z}})\big)\,T_{0,K}(\underline{u}-\overline{u})\,\mathrm{d}x\mathrm{d}t \\
&\quad -j\int_{t_0}^{\tau}\int_{1/j}^{2/j}\big((\underline{\mathbf{z}}-\alpha_1)+(\alpha_1-\alpha_2)+(\alpha_2-\overline{\mathbf{z}})\big)\,T_{0,K}(\underline{u}-\overline{u})\,\mathrm{d}x\mathrm{d}t \\
&\le jK\int_{t_0}^{\tau}\int_{L-2/j}^{L-1/j}\left\{[\underline{\mathbf{z}}-\beta_1]^+ + [\overline{\mathbf{z}}-\beta_2]^-\right\}\mathrm{d}x\mathrm{d}t \\
&\quad +jK\int_{t_0}^{\tau}\int_{1/j}^{2/j}\left\{[\underline{\mathbf{z}}-\alpha_1]^- + [\overline{\mathbf{z}}-\alpha_2]^+\right\}\mathrm{d}x\mathrm{d}t\,. \qquad (6.2)
\end{aligned}$$

Let us address the first term in the right-hand side of (6.1). To this aim, let $\Omega_j \subset\subset \Omega$ be any open set such that $\operatorname{supp}\rho_j \subset \Omega_j$. Then $\underline{u},\ \overline{u} \in L^1_w(0,T;BV(\Omega_j))$, and by [2, Corollary 3.1] there holds

$$\big(T_{0,K}(\underline{u}(t)-\overline{u}(t))\big)_x = \chi_{\{0<\underline{u}(\cdot,t)-\overline{u}(\cdot,t)<K\}}(\underline{u}(t)-\overline{u}(t))_x \quad a.e.\ \text{in}\ \Omega_j\,, \qquad (6.3)$$

$$D^s_\pm T_{0,K}(\underline{u}(t)-\overline{u}(t)) \le D^s_\pm(\underline{u}(t)-\overline{u}(t)) \quad \text{in}\ \mathcal{M}(\Omega_j) \qquad (6.4)$$

for every $K>0$ and for $a.e.\ t\in(0,T)$.

By (6.3) and the nondecreasing character of the map $\xi \mapsto \mathbf{a}(z,\xi)$, we get

$$\begin{aligned}
&-\int_{Q_{t_0,\tau}}(\underline{\mathbf{z}}-\overline{\mathbf{z}})\big(T_{0,K}(\underline{u}-\overline{u})\big)_x\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \\
&= -\int_{Q_{t_0,\tau}\cap\{0<\underline{u}-\overline{u}<K\}}(\mathbf{a}(\underline{u},\underline{u}_x)-\mathbf{a}(\underline{u},\overline{u}_x))(\underline{u}-\overline{u})_x\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \\
&\quad -\int_{Q_{t_0,\tau}\cap\{0<\underline{u}-\overline{u}<K\}}(\mathbf{a}(\underline{u},\overline{u}_x)-\mathbf{a}(\overline{u},\overline{u}_x))(\underline{u}-\overline{u})_x\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \\
&\le \int_{Q_{t_0,\tau}\cap\{0<\underline{u}-\overline{u}<K\}}|\mathbf{a}(\underline{u},\overline{u}_x)-\mathbf{a}(\overline{u},\overline{u}_x)|\,|(\underline{u}-\overline{u})_x|\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \\
&\le L_0K\int_{Q_{t_0,\tau}\cap\{0<\underline{u}-\overline{u}<K\}}|(\underline{u}-\overline{u})_x\,|\,\rho_j(x)\,\mathrm{d}x\mathrm{d}t \qquad (6.5)
\end{aligned}$$

(here $L_0>0$ is the Lipschitz constant of $\mathbf{a}$).

Next, observe that by Proposition 3.1, and since $-1 \le \underline{\mathbf{z}}(\mathbf{t}), \overline{\mathbf{z}}(\mathbf{t}) \le 1$ in Ω, for every nonnegative $\zeta \in C_c^1(\Omega_j)$ there holds

$$\begin{aligned}
&\int_{\Omega_j} \big(\underline{\mathbf{z}}(\mathbf{t}) - \overline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s(\underline{u}(t) - \overline{u}(t)) \\
&\quad = \int_{\Omega_j} \big(\underline{\mathbf{z}}(\mathbf{t}) - \overline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s\underline{u}(t) + \int_{\Omega_j} \big(\overline{\mathbf{z}}(\mathbf{t}) - \underline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s\overline{u}(t) \\
&\quad = \int_{\Omega_j} \big(1 - \overline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s_+\underline{u}(t) + \int_{\Omega_j} \big(1 + \overline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s_-\underline{u}(t) \\
&\qquad + \int_{\Omega_j} \big(1 - \underline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s_+\overline{u}(t) + \int_{\Omega_j} \big(1 + \underline{\mathbf{z}}(t)\big)\, \zeta(x)\, dD^s_-\overline{u}(t) \ge 0
\end{aligned}$$

(for $a.e.\ t \in (0, T)$). In view of the arbitrariness of ζ, this implies that

$$\begin{aligned}
&\underline{\mathbf{z}}(\mathbf{t}) - \overline{\mathbf{z}}(t) \ge 0 \quad D^s_+(\underline{u}(t) - \overline{u}(t))\text{-}a.e. \text{ in } \Omega_j\,, \\
&\underline{\mathbf{z}}(\mathbf{t}) - \overline{\mathbf{z}}(t) \le 0 \quad D^s_-(\underline{u}(t) - \overline{u}(t))\text{-}a.e. \text{ in } \Omega_j\,.
\end{aligned}$$

Notice that, by (6.4), the above inequalities hold true $D^s_+ T_{0,K}(\underline{u}(t) - \overline{u}(t))$-$a.e.$ and $D^s_- T_{0,K}(\underline{u}(t) - \overline{u}(t))$-$a.e.$ in Ω_j, respectively. Thus, it follows that

$$-\int_{\Omega} \big(\underline{\mathbf{z}}(\mathbf{t}) - \overline{\mathbf{z}}(t)\big)\, \rho_j(x)\, dD^s T_{0,K}(\underline{u}(t) - \overline{u}(t)) \le 0\,. \tag{6.6}$$

Combining (6.2), (6.5) and (6.6) with (6.1) we get

$$\begin{aligned}
&\int_{Q_{t_0,\tau}} (\underline{u} - \overline{u})_t\, T_{0,K}(\underline{u} - \overline{u})\, \rho_j(x)\, \mathrm{d}x\mathrm{d}t \\
&\quad \le jK \int_{t_0}^{\tau}\int_{L-2/j}^{L-1/j} \left\{[\underline{\mathbf{z}} - \beta_1]^+ + [\overline{\mathbf{z}} - \beta_2]^-\right\} \mathrm{d}x\mathrm{d}t \\
&\qquad + jK \int_{t_0}^{\tau}\int_{1/j}^{2/j} \left\{[\underline{\mathbf{z}} - \alpha_1]^- + [\overline{\mathbf{z}} - \alpha_2]^+\right\} \mathrm{d}x\mathrm{d}t \\
&\qquad + L_0 K \int_{Q_{t_0,\tau} \cap \{0 < \underline{u} - \overline{u} < K\}} |(\underline{u} - \overline{u})_x|\, \rho_j(x)\, \mathrm{d}x\mathrm{d}t\,.
\end{aligned}$$

Dividing both sides of the above inequality by K and letting $K \to 0^+$, we obtain

$$\begin{aligned}
&\int_{\Omega} [\underline{u}(\tau) - \overline{u}(\tau)]^+\, \rho_j(x)\, \mathrm{d}x - \int_{\Omega} [\underline{u}(t_0) - \overline{u}(t_0)]^+\, \rho_j(x)\, \mathrm{d}x \\
&\quad \le j \int_{t_0}^{\tau}\int_{L-2/j}^{L-1/j} \left\{[\underline{\mathbf{z}} - \beta_1]^+ + [\overline{\mathbf{z}} - \beta_2]^-\right\} \mathrm{d}x\mathrm{d}t \\
&\qquad + j \int_{t_0}^{\tau}\int_{1/j}^{2/j} \left\{[\underline{\mathbf{z}} - \alpha_1]^- + [\overline{\mathbf{z}} - \alpha_2]^+\right\} \mathrm{d}x\mathrm{d}t\,.
\end{aligned} \tag{6.7}$$

Since $\underline{\mathbf{z}},\ \overline{\mathbf{z}} \in L^2(t_0, \tau; H^1(\Omega))$, $|\underline{\mathbf{z}}|\,, |\overline{\mathbf{z}}| \le 1$ and

$$\underline{\mathbf{z}}(0, t) \ge \alpha_1 \ge \alpha_2 \ge \overline{\mathbf{z}}(0, t)\,, \qquad \underline{\mathbf{z}}(L, t) \le \beta_1 \le \beta_2 \le \overline{\mathbf{z}}(L, t)\,,$$

letting $j \to \infty$ in (6.7) plainly gives $\int_\Omega [\underline{u}(\tau) - \overline{u}(\tau)]^+ \, dx \le \int_\Omega [\underline{u}(t_0) - \overline{u}(t_0)]^+ \, dx$. Therefore the conclusion follows taking the limit as $t_0 \to 0^+$ in this inequality, as $\underline{u}, \overline{u} \in C([0,T]; L^1(\Omega))$ and $\underline{u}(0) \le \underline{u}_0, \overline{u}_0 \le \overline{u}(0)$ $a.e.$ in Ω. □

6.2. Existence for L^2-initial data

Let $\{\varepsilon_k\}$ be any sequence such that $\varepsilon_k \to 0^+$ as $k \to \infty$. For every $u_0 \in L^2(\Omega)$ let $\{u_{0k}\} \subseteq C^1(\overline{\Omega})$ satisfy

$$\begin{cases} u_{0k} \to u_0 \quad \text{in } L^2(\Omega), \\ \varepsilon_k \int_\Omega \left([u_{0k}]_x\right)^2 dx \to 0 \end{cases} \tag{6.8}$$

(such a sequence $\{u_{0k}\}$ can be constructed by convolution). For every $\varepsilon_k > 0$, we denote by u_k the solution to (4.1)–(4.2) with $\varepsilon = \varepsilon_k$ and initial datum u_{0k}, and we set

$$v_k = \mathbf{a}(u_k, u_{kx}) + \sqrt{\varepsilon_k}\, u_{kx} + \varepsilon_k\, u_{kxt}\,. \tag{6.9}$$

The following propositions rely on the a priori estimates in Propositions 5.2, 5.3 and 5.4.

Proposition 6.1. *There holds*

$$\int_{Q_{0,T}} \left(v_k - \boldsymbol{a}(u_k, u_{kx})\right) \zeta \, dx dt \to 0 \quad \textit{for every } \zeta \in C^1([0,T]; C(\overline{\Omega})) \tag{6.10}$$

and, for every $\tau \in (0,T)$,

$$\|v_k - \boldsymbol{a}(u_k, u_{kx})\|_{L^2(Q_{\tau,T})} = \left\|\sqrt{\varepsilon_k}\, u_{kx} + \varepsilon_k\, u_{kxt}\right\|_{L^2(Q_{\tau,T})} \to 0\,, \tag{6.11}$$

$$\int_{Q_{\tau,T}} \varepsilon_k |u_{kxt}\, u_{kx}| \, dx dt \to 0\,. \tag{6.12}$$

Proof. Multiplying the equation $v_k - \mathbf{a}(u_k, u_{kx}) = \sqrt{\varepsilon_k} u_{kx} + \varepsilon_k u_{kxt}$ by the test function $\zeta \in C^1([0,T]; C(\overline{\Omega}))$ and integrating by parts gives

$$\begin{aligned} \int_{Q_{0,T}} \left(v_k - \mathbf{a}(u_k, u_{kx})\right) \zeta \, dx dt &= \sqrt{\varepsilon_k} \int_{Q_{0,T}} u_{kx}\, \zeta \, dx dt + \varepsilon_k \int_\Omega u_{kx}(T)\, \zeta(T) \, dx \\ &\quad - \varepsilon_k \int_\Omega [u_{0k}]_x\, \zeta(0) \, dx - \varepsilon_k \int_{Q_{0,T}} u_{kx}\, \zeta_t dx dt\,, \end{aligned}$$

and the right-hand side of the above equality converges to zero as $k \to \infty$ by (5.11) and (6.8).

By (5.13) and (6.8) there exists $C > 0$ independent of ε_k such that

$$\varepsilon_k \int_{Q_{0,T}} t \left(u_{kxt}\right)^2 dx dt \le C \quad \text{and} \quad \sqrt{\varepsilon_k} \int_\Omega \left(u_{kx}\right)^2(t) \, dx \le \frac{C}{t} \quad \text{for all } t \in (0,T)\,.$$

Hence, for every $\tau \in (0,T)$, we have

$$\varepsilon_k \int_{Q_{\tau,T}} \left(u_{kxt}\right)^2 dx dt \le \frac{C}{\tau}, \qquad \sqrt{\varepsilon_k} \int_{Q_{\tau,T}} \left(u_{kx}\right)^2 dx dt \le \frac{CT}{\tau}\,, \tag{6.13}$$

which plainly gives (6.11). Finally, in order to prove (6.12), let us observe that for every $\theta \in (0,1)$ there holds

$$\varepsilon_k \int_{Q_{\tau,T}} |u_{kxt}\, u_{kx}|\, \mathrm{d}x\mathrm{d}t \le \frac{\varepsilon_k^{(1+\theta)/2}}{2} \int_{Q_{\tau,T}} \left(u_{kx}\right)^2 \mathrm{d}x\mathrm{d}t + \frac{\varepsilon_k^{(3-\theta)/2}}{2} \int_{Q_{\tau,T}} \left(u_{kxt}\right)^2 \mathrm{d}x\mathrm{d}t \to 0\,,$$

the latter convergence in the above inequality being a direct consequence of (6.13). □

Proposition 6.2. *There exists* $u \in C([0,T]; L^2(\Omega)) \cap L^1_w(0,T; BV_{\mathrm{loc}}(\Omega))$, $u(0) = u_0$ *a.e. in* Ω, *such that for every* $\tau \in (0,T)$ *and* $a, b \in \mathbb{R}$, $a < b$,

$$u_t \in L^2(Q_{\tau,T})\,, \quad u \in L^\infty_w(\tau,T; BV_{\mathrm{loc}}(\Omega)), \quad T_{a,b}(u) \in L^1_w(\tau,T; BV(\Omega))\,.$$

Moreover, possibly up to a subsequence (not relabeled), there holds

$$u_{kt} \rightharpoonup u_t \quad \text{in } L^2(Q_{\tau,T}) \quad \text{for all } \tau \in (0,T)\,, \tag{6.14}$$

$$u_k(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega) \quad \text{for every } t \in (0,T]\,, \tag{6.15}$$

and for a.e. $t \in (0,T)$

$$u_k(t) \to u(t) \quad \text{in } L^p(\Omega) \quad \text{for every} \quad 1 \le p < 2\,, \tag{6.16}$$

$$\int_\Omega u_{kx}(t)\, \rho\, \mathrm{d}x \to \int_\Omega \rho\, dDu(t) \quad \text{for all } \rho \in C_c(\Omega)\,. \tag{6.17}$$

Remark 6.1.

(i) By (5.8), (6.16) and the Dominated Convergence theorem

$$u_k \to u \quad \text{in } L^p(Q_{0,T}) \quad \text{for all } 1 \le p < 2\,. \tag{6.18}$$

(ii) By (5.24), for every $\Omega' \subset\subset \Omega$ and $\tau \in (0,T)$, there exists $C_{\Omega',\tau} > 0$ independent of k such that

$$\|u_{kx}\|_{L^\infty(\tau,T;L^1(\Omega'))} \le C_{\Omega',\tau}\,, \tag{6.19}$$

$$\|u_k\|_{L^\infty(Q'_{\tau,T})} \le C_{\Omega',\tau} \tag{6.20}$$

(recall that Ω is one dimensional and see also (5.8)), whence

$$u_k \to u \quad \text{in } L^p(Q'_{\tau,T}) \quad \text{for all } 1 \le p < \infty\,, \tag{6.21}$$

since by (6.18), up to a subsequence (not relabeled), we may assume that $u_k \to u$ *a.e.* in $Q_{0,T}$.

(iii) By (6.19), (6.17) and the Dominated Convergence theorem, we get

$$\int_{Q_{0,T}} \zeta\, u_{kx}\, \mathrm{d}x\mathrm{d}t \to \int_0^T \left(\int_\Omega \zeta(t)\, dDu(t) \right) \mathrm{d}t \quad \text{for all } \zeta \in C_c(Q_{0,T})\,. \tag{6.22}$$

Let us also explicitly observe that the limiting function u given in Proposition 6.2 belongs to $L^\infty(Q'_{\tau,T})$ for every Ω' and τ as above.

Proof of Proposition 6.2. In view of (5.8), (5.14) and (5.23), the sequence $\{u_k\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, $\{u_{kt}\}$ is bounded in $L^2(Q_{\tau,T})$ for every $\tau\in(0,T)$ and $\{u_{kx}\}$ is bounded in $L^1(Q'_{0,T})$ for every $\Omega'\subset\subset\Omega$. Then there exists $u\in L^\infty(0,T;L^2(\Omega))\cap BV_{\mathrm{loc}}(Q_{0,T})$, with $u_t\in L^2(Q_{\tau,T})$ for every $\tau\in(0,T)$ (whence $u\in C((0,T];L^2(\Omega))$ by the arbitrariness of τ) such that, possibly up to a subsequence not relabeled,

$$u_k\rightharpoonup u\quad\text{in } L^2(Q_{0,T})\,. \tag{6.23}$$

Moreover, for every $\tau\in(0,T)$ and $\Omega'\subset\subset\Omega$ there exists a subsequence (not relabeled), possibly depending on Ω' and τ, such that $u_k\to u$ *a.e.* in $Q'_{\tau,T}$ and in $L^1(Q'_{\tau,T})$ (this follows as $\{u_k\}$ is bounded in $BV(Q'_{\tau,T})$), and the convergence in (6.14) is satisfied. Hence, choosing $\Omega'=\Omega_p=(1/p,L-1/p)$ and $\tau=1/p$ ($p\in\mathbb{N}$), by a diagonal argument there exists a subsequence of $\{u_k\}$ (again not relabeled) along which, for every $\tau\in(0,T)$ and $\Omega'\subset\subset\Omega$,

$$u_k\to u\quad a.e.\ \text{in } Q_{0,T}\,,\qquad u_k\to u\ \text{in } L^1(Q'_{\tau,T}) \tag{6.24}$$

and $u_{kt}\rightharpoonup u_t$ in $L^2(Q_{\tau,T})$. In view of (6.24), there exists a subset $N\subset(0,T)$ of zero Lebesgue measure such that, for every $t\in(0,T)\setminus N$,

$$u_k(t)\to u(t)\quad a.e.\ \text{in }\Omega\ \text{and in } L^1(\Omega') \tag{6.25}$$

for all Ω' as above. Then, the claims $u\in L^\infty_w(\tau,T;BV_{\mathrm{loc}}(\Omega))\cap L^1_w(0,T;BV_{\mathrm{loc}}(\Omega))$ and $T_{a,b}(u)\in L^1_w(\tau,T;BV(\Omega))$ (for all $\tau\in(0,T)$ and $a<b$) follow from the above convergences, combined with (5.23), (5.24) and (5.27), respectively (see also [6, Lemma 5]).

From (6.25) and the a priori estimate (5.8), we get (6.15) for every $t\in(0,T)\setminus N$; actually, this convergence holds true for every $t\in(0,T]$ since $u\in C((0,T];L^2(\Omega))$ and the sequences $\{u_k\}$, $\{u_{kt}\}$ are bounded respectively in $L^\infty(0,T;L^2(\Omega))$ and $L^2(Q_{\tau,T})$ for every $\tau\in(0,T)$. Therefore (6.16) follows from (6.15) and (6.25), for every $t\in(0,T)\setminus N$ and $1\le p<2$. Analogously, to prove the convergence in (6.17) for every $t\in(0,T)\setminus N$, it is enough to combine (6.16) and the a priori estimate (5.24).

It only remains to prove that

$$u(t)\to u_0\quad\text{in } L^2(\Omega)\quad\text{as } t\to 0^+\,. \tag{6.26}$$

To this aim, observe that since $|\mathbf{a}(u_k,u_{kx})|\le 1$ (see (1.1)), there exists $\mathbf{z}\in L^\infty(Q_{0,T})$ such that (possibly up to a subsequence not relabeled)

$$\mathbf{a}(u_k,u_{kx})\stackrel{*}{\rightharpoonup}\mathbf{z}\quad\text{in } L^\infty(Q_{0,T})\,, \tag{6.27}$$

whence (see (6.10))

$$\int_{Q_{0,T}} v_k\, \zeta \,\mathrm{d}x\mathrm{d}t \to \int_{Q_{0,T}} \mathbf{z}\, \zeta \,\mathrm{d}x\mathrm{d}t \quad \text{for every } \zeta \in C^1([0,T];C(\overline{\Omega}))\,. \tag{6.28}$$

For every $\zeta \in C^\infty(\overline{Q}_{0,T})$, $\zeta(T)=0$ in Ω and $\zeta(t) \in C_c^\infty(\Omega)$ for every $t \in [0,T]$, letting $k \to \infty$ in the weak formulation of problems (4.1)–(4.2), namely

$$\int_{Q_{0,T}} u_k\, \zeta_t \,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,T}} v_k\, \zeta_x \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} F\, \zeta \,\mathrm{d}x\mathrm{d}t = -\int_\Omega u_{0k}\, \zeta(x,0)\,\mathrm{d}x\,,$$

gives

$$\int_{Q_{0,T}} u\, \zeta_t \,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,T}} \mathbf{z}\, \zeta_x \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} F\, \zeta \,\mathrm{d}x\mathrm{d}t = -\int_\Omega u_0\, \zeta(x,0)\,\mathrm{d}x \tag{6.29}$$

(here we have used (6.8), (6.23) and (6.28)). For every $t_0 \in (0,T)$, by standard approximation arguments we can choose in (6.29) $\zeta(x,t) = \rho(x)\, h_j(t)$ with $\rho \in C_c^1(\Omega)$ and

$$h_j(t) = \chi_{[0,t_0]}(t) + j\left(t_0 + \frac{1}{j} - t\right)\chi_{(t_0,t_0+1/j)}(t) \qquad (j \in \mathbb{N})\,.$$

Hence, letting $j \to \infty$ in (6.29) with this choice of ζ, we obtain (recall that $u \in C((0,T];L^2(\Omega))$)

$$\int_\Omega u(t_0)\, \rho\,\mathrm{d}x - \int_\Omega u_0 \rho\,\mathrm{d}x = -\int_{Q_{0,t_0}} \mathbf{z}\, \rho' \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,t_0}} F\, \rho\,\mathrm{d}x\mathrm{d}t.$$

Let $t_n \to 0^+$ be fixed arbitrarily. By last equality with $t_0 = t_n$, it follows that $u(t_n) \to u_0$ in $\mathcal{D}'(\Omega)$, hence

$$u(t_n) \rightharpoonup u_0 \quad \text{in } L^2(\Omega) \tag{6.30}$$

since $u \in L^\infty(0,T;L^2(\Omega))$. This plainly gives

$$\int_\Omega u_0^2\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega u^2(t_n)\,\mathrm{d}x$$

and, by the arbitrariness of $t_n \to 0^+$, (6.26) will follow from the above inequality and (6.30), if we prove that

$$\limsup_{n\to\infty} \int_\Omega u^2(t_n)\,\mathrm{d}x \le \int_\Omega u_0^2\,\mathrm{d}x\,. \tag{6.31}$$

This inequality is obtained by taking the limit as $k \to \infty$ in (5.9). This yields

$$\frac{1}{2}\int_\Omega u^2(\tau)\,\mathrm{d}x \le \tau D_{1,T}\left[1 + \int_\Omega u_0^2\mathrm{d}x\right] + \frac{1}{2}\int_\Omega u_0^2\,\mathrm{d}x$$

(see (6.8) and (6.15)). Choosing $\tau = t_n$ in the above inequality and letting $n \to \infty$, we obtain (6.31). □

Proposition 6.3. *Let u be the limiting function as given by Proposition 6.2. Then, $\boldsymbol{a}(u,u_x) \in L^2(\tau,T;H^1(\Omega))$ for every $\tau \in (0,T)$ and, possibly up to a subsequence of $\{u_k\}$ not relabeled,*

$$\boldsymbol{a}(u_k,u_{kx}) \overset{*}{\rightharpoonup} \boldsymbol{a}(u,u_x) \quad in \ L^\infty(Q_{0,T})\,, \tag{6.32}$$

$$v_k \rightharpoonup \boldsymbol{a}(u,u_x) \ \ in \ L^2(\tau,T;H^1(\Omega)) \ \ for\ every\ \tau \in (0,T)\,. \tag{6.33}$$

Moreover, for $a.e.\ t \in (0,T)$ there holds

$$u_t(t) = \partial_x\big(\boldsymbol{a}(u(t),u_x(t))\big) + F(t) \quad a.e.\ in\ \Omega\,, \tag{6.34}$$

$$[\boldsymbol{a}(u(t),u_x(t))](0) = \alpha\,, \quad [\boldsymbol{a}(u(t),u_x(t))](L) = \beta\,. \tag{6.35}$$

Proof. Let $\mathbf{z} \in L^\infty(Q_{0,T})$ be the function given in (6.27). Combining (6.28) with the a priori estimate

$$\sup_k \|v_k - r_{\alpha,\beta}\|^2_{L^2(\tau,T;H^1_0(\Omega))} \le C/\tau \qquad (\tau \in (0,T))$$

(see (5.13)) plainly gives $\mathbf{z} \in L^2(\tau,T;H^1(\Omega))$ for every τ as above,

$$\mathbf{z}(0,t) = \alpha, \qquad \mathbf{z}(L,t) = \beta \quad \text{for } a.e.\ t \in (0,T) \tag{6.36}$$

and, possibly up to a subsequence not relabeled,

$$v_k \rightharpoonup \mathbf{z} \quad \text{in } L^2(\tau,T;H^1(\Omega))\,. \tag{6.37}$$

In view of (6.14) and (6.37), letting $k \to \infty$ in the equality $u_{kt} = v_{kx} + F$, we get for every $\zeta \in C_c(Q_{0,T})$ the equality $\int_{Q_{0,T}} u_t\,\zeta\,\mathrm{d}x\mathrm{d}t = \int_{Q_{0,T}} \mathbf{z}_x\,\zeta\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} F\,\zeta\,\mathrm{d}x\mathrm{d}t$, whence

$$u_t(t) = \mathbf{z}_x(t) + F(t) \quad a.e.\ \text{in}\ \Omega \tag{6.38}$$

for $a.e.\ t \in (0,T)$ by the arbitrariness of ζ. Therefore the conclusion follows from (6.27) and (6.36)–(6.38), since arguing as in [22, Propositions 6.3 and 6.4] it can be seen that $\mathbf{z} = \mathbf{a}(u,u_x)$ $a.e.$ in $Q_{0,T}$. □

Proposition 6.4. *Let u be the limiting function given by Proposition 6.2.*

(i) *For every $a,b \in \mathbb{R}$, $a<b$, there exists a null set $N_{a,b} \subset (0,T)$ such that for all $t \in (0,T)\setminus N_{a,b}$ and for every $\rho \in C_c(\Omega)$ there holds*

$$\int_\Omega \rho\,\boldsymbol{a}(u(t),u_x(t))\,dDT_{a,b}(u(t)) = \int_\Omega \rho\,dh(u(t),DT_{a,b}(u(t)))\,. \tag{6.39}$$

(ii) *For $a.e.\ t \in (0,T)$ and for every $\rho \in C_c(\Omega)$ there holds*

$$\int_\Omega \rho\,\boldsymbol{a}(u(t),u_x(t))\,dDu(t) = \int_\Omega \rho\,dh(u(t),Du(t))\,. \tag{6.40}$$

(iii) *For every* $\zeta \in C^1_c(Q_{0,T})$ *there holds*

$$\lim_{k\to\infty} \int_{Q_{0,T}} \boldsymbol{a}(u_k, u_{kx})\, u_{kx}\, \zeta \,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega \zeta(t)\, dh(u(t), Du(t)) \right) \mathrm{d}t \,, \tag{6.41}$$

$$\lim_{k\to\infty} \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left(u_{kx}\right)^2 \zeta \,\mathrm{d}x\mathrm{d}t = 0\,, \tag{6.42}$$

where $\{u_k\}$ *is any sequence along which all the convergences in Propositions 6.2–6.3 are satisfied.*

Remark 6.2. Since $u \in L^\infty_w(\tau, T; BV_{\mathrm{loc}}(\Omega))$ and $T_{a,b}(u) \in L^1_w(\tau, T; BV(\Omega))$, whence

$$|Du| \in L^\infty_{w*}(\tau, T; \mathcal{M}_{\mathrm{loc}}(\Omega))\,, \quad |DT_{a,b}(u)| \in L^1_{w*}(\tau, T; \mathcal{M}(\Omega))$$

for every $\tau \in (0, T)$, by the definition of the measures $h(u, Du)$ and $h(u, DT_{a,b}(u))$ in (2.3), (2.5), it follows that

$$h(u, Du) \in L^\infty_{w*}(\tau, T; \mathcal{M}_{\mathrm{loc}}(\Omega))\,, \quad h(u, DT_{a,b}(u)) \in L^1_{w*}(\tau, T; \mathcal{M}(\Omega))$$

for all $\tau \in (0, T)$ and $a, b \in \mathbb{R}$, $a < b$. Moreover we explicitly notice that (6.39) plainly implies item (iv) in Definition 3.1.

Proof of Proposition 6.4. Let us preliminary notice that it is enough to prove (6.39), (6.40) and (6.41) only for nonnegative $\rho \in C_c(\Omega)$ and nonnegative $\zeta \in C^1_c(Q_{0,T})$, respectively.

(i) For every nonnegative $\eta \in C^1_c(0, T)$ and $\rho \in C^1_c(\Omega)$, $\rho \geq 0$, by (4.9) we get

$$\begin{aligned} &\int_{Q_{0,T}} h(u_k, [T_{a,b}(u_k)]_x)\, \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t + \varepsilon_k \int_{Q_{0,T}} u_{kxt}\, [T_{a,b}(u_k)]_x\, \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t \\ &\quad + \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left([T_{a,b}(u_k)]_x\right)^2 \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} v_k T_{a,b}(u_k)\, \rho'(x)\, \eta(t) \,\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_{0,T}} \left\{ J_{a,b}(u_k)\, \eta'(t)\, \rho(x) + F\, T_{a,b}(u_k)\, \eta(t)\, \rho(x) \right\} \mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{6.43}$$

In order to take the limit as $k \to \infty$ in the above equality, we firstly observe that by (6.12), (6.21) and (6.33) there holds

$$\lim_{k\to\infty} \varepsilon_k \int_{Q_{0,T}} u_{kxt}\, [T_{a,b}(u_k)]_x\, \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t = 0\,,$$

$$\lim_{k\to\infty} \int_{Q_{0,T}} v_k T_{a,b}(u_k)\, \rho'(x)\, \eta(t) \,\mathrm{d}x\mathrm{d}t = \int_{Q_{0,T}} \mathbf{a}(u, u_x) T_{a,b}(u)\, \rho'(x)\, \eta(t) \,\mathrm{d}x\mathrm{d}t\,,$$

$$\lim_{k\to\infty}\int_{Q_{0,T}}\left\{J_{a,b}(u_k)\,\eta'(t)\,\rho(x)+F\,T_{a,b}(u_k)\,\eta(t)\,\rho(x)\right\}\,\mathrm{d}x\mathrm{d}t$$
$$=\int_{Q_{0,T}}\left\{J_{a,b}(u)\,\eta'(t)\,\rho(x)+F\,T_{a,b}(u)\,\eta(t)\,\rho(x)\right\}\,\mathrm{d}x\mathrm{d}t\,,$$

whence

$$\begin{aligned}
&\liminf_{k\to\infty}\int_{Q_{0,T}}h(u_k,[T_{a,b}(u_k)]_x)\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\liminf_{k\to\infty}\sqrt{\varepsilon_k}\int_{Q_{0,T}}\left([T_{a,b}(u_k)]_x\right)^2\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\
&\le\int_{Q_{0,T}}\left\{J_{a,b}(u)\,\eta'(t)\,\rho(x)+F\,T_{a,b}(u)\,\eta(t)\,\rho(x)\right\}\,\mathrm{d}x\mathrm{d}t\\
&\quad-\int_{Q_{0,T}}\mathbf{a}(u,u_x)\,T_{a,b}(u)\,\rho'(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t\\
&=-\int_{Q_{0,T}}\left\{T_{a,b}(u)\,u_t-F\,T_{a,b}(u)-[\mathbf{a}(u,u_x)]_x\,T_{a,b}(u)\right\}\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^T\left(\int_\Omega\mathbf{a}(u(t),u_x(t))\,\rho\,dDT_{a,b}(u(t))\right)\,\eta(t)\,\mathrm{d}t\\
&=\int_0^T\left(\int_\Omega\mathbf{a}(u(t),u_x(t))\,\rho\,dDT_{a,b}(u(t))\right)\,\eta(t)\,\mathrm{d}t
\end{aligned}\tag{6.44}$$

(here we have used (6.34)). Moreover, relying on (1.5), (1.6) and the assumption $\mathbf{a}(z,0)=0$ (see (H_1)), for $a.e.\ t\in(0,T)$ we obtain (below we omit the dependence on t for notational simplicity)

$$\begin{aligned}
&h(u_k,[T_{a,b}(u_k)]_x)\\
&\quad=[T_{a,b}(u_k)]_x\,\mathbf{a}(u_k,[T_{a,b}(u_k)]_x)\\
&\quad=[T_{a,b}(u_k)]_x\,\mathbf{a}(T_{a,b}(u_k),[T_{a,b}(u_k)]_x)\ge[T_{a,b}(u)]_x\,\mathbf{a}(T_{a,b}(u_k),[T_{a,b}(u_k)]_x)\\
&\qquad+f(T_{a,b}(u_k),[T_{a,b}(u_k)]_x)-f(T_{a,b}(u_k),[T_{a,b}(u)]_x)\\
&\quad=[T_{a,b}(u)]_x\,\mathbf{a}(u_k,u_{kx})\,\chi_{\{a<u_k<b\}}\\
&\qquad+f(T_{a,b}(u_k),[T_{a,b}(u_k))]_x)-f(T_{a,b}(u_k),[T_{a,b}(u))]_x)
\end{aligned}$$

Combining the above inequality with Lemma 2.2 (recall that $\eta,\rho\ge0$), (6.16), (6.18) (see also (6.24)) and (6.32) plainly gives

$$\begin{aligned}
&\liminf_{k\to\infty}\int_{Q_{0,T}}h(u_k,[T_{a,b}(u_k)]_x)\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\
&\quad\ge\int_{Q_{0,T}}[T_{a,b}(u)]_x\,\mathbf{a}(u,u_x)\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{Q_{0,T}}f\left(T_{a,b}(u),[T_{a,b}(u)]_x\right)\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$+\int_0^T \left(\int_\Omega \rho(x)\, df(T_{a,b}(u(t)), DT_{a,b}(u(t)))\right) \eta(t)\,\mathrm{d}t$$
$$= \int_0^T \left(\int_\Omega \rho(x)\, dh(u(t), DT_{a,b}(u(t)))\right) \eta(t)\,\mathrm{d}t \tag{6.45}$$

(see (2.3)–(2.5)). By (6.44), (6.45) and the arbitrariness of $\eta \in C_c^1(0,T)$, $\eta \geq 0$, for every $a, b \in \mathbb{R}$, $a < b$, there exists a null set $N_{a,b} \subset (0,T)$ such that for all $t \in (0,T) \setminus N_{a,b}$ and for every nonnegative $\rho \in C_c(\Omega)$ there holds

$$\int_\Omega \rho\, dh\big(u(t), DT_{a,b}(u(t))\big) \leq \int_\Omega \rho\, \mathbf{a}(u(t), u_x(t))\, dDT_{a,b}(u(t)) \tag{6.46}$$

(the choice of the set $N_{a,b}$ can be made independent of ρ by separability arguments). Therefore claim (i) follows from (6.46), as by (1.1) and (2.5) the reverse inequality is straightforward:

$$\begin{aligned}&\int_\Omega \rho\, \mathbf{a}(u(t), u_x(t))\, dDT_{a,b}(u(t))\\ &\quad\leq \int_\Omega \rho\, \mathbf{a}(u(t), u_x(t))\, [T_{a,b}(u(t))]_x\,\mathrm{d}x + \int_\Omega \rho\, d|D^s T_{a,b}(u(t))|\\ &\quad= \int_\Omega \rho\, dh(u(t), DT_{a,b}(u(t)))\,.\end{aligned}$$

(ii) The proof is essentially the same as the one of item (i). The only difference is that we have to take $\Omega' \subset\subset \Omega$ and $\tau \in (0,T)$ such that $\operatorname{supp}\rho \subseteq \Omega'$ and $\operatorname{supp}\eta \subseteq (\tau, T)$. Therefore, after repeating the same computations we obtain

$$\begin{aligned}&\int_0^T \left(\int_\Omega \rho(x)\, dh(u(t), Du(t))\right) \eta(t)\,\mathrm{d}t\\ &\quad\leq \liminf_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\ &\quad\leq \limsup_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\\ &\quad\leq \limsup_{k\to\infty} \left\{\int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon_k}\int_{Q_{0,T}} \left(u_{kx}\right)^2 \eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t\right\}\\ &\quad\leq \int_0^T \left(\int_\Omega \mathbf{a}(u(t), u_x(t))\,\rho\, dDu(t)\right) \eta(t)\,\mathrm{d}t\\ &\quad= \int_0^T \left(\int_\Omega \rho(x)\, dh(u(t), Du(t))\right) \eta(t)\,\mathrm{d}t\,,\end{aligned} \tag{6.47}$$

for every nonnegative $\eta \in C_c^1(0,T)$ and $\rho \in C_c^1(\Omega)$, $\rho \geq 0$. From here, the proof finishes as in (i).

(iii) From (6.47), we get

$$\lim_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega \rho(x)\, dh(u(t), Du(t))\right) \eta(t)\,\mathrm{d}t$$

and

$$\limsup_{k\to\infty} \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left(u_{kx}\right)^2 \eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t = 0.$$

Therefore (6.41) and (6.42) follow from the above equalities, as every $\zeta \in C^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ can be uniformly approximated in bounded subsets of $\mathbb{R}^2$ by finite sums $\sum_{i=1}^p f^{i,p}(x) g^{i,p}(t)$ with $f^{i,p}$, $g^{i,p}$ bounded and C^1-functions ($1 \le i \le p$; *e.g.*, see [19, Théorème D.1.1]). □

Proof of Theorem 3.3. For every $u_0 \in L^2(\Omega)$, the existence of a strong solution u to problem $(P^{u_0}_{\alpha,\beta})$, $u \in C([0,T]; L^2(\Omega)) \cap L^\infty(\tau, T; BV_{\mathrm{loc}}(\Omega))$ for every $\tau \in (0,T)$, is ensured by Propositions 6.2, 6.3 and 6.4. Moreover, if (H_2) is satisfied with $\max\{|\alpha|, |\beta|\} < C_0 \le 1$, from the a priori estimates in Proposition 5.3-(ii), it follows that $u \in L^1_w(0,T; BV(\Omega)) \cap L^\infty_w(\tau, T; BV(\Omega))$ for all $\tau \in (0,T)$. □

Proof of Proposition 3.5. Let $\{u_k\}$ and $\{v_k\}$ be sequences along which all the convergences in Sect. 6.2 are satisfied. From the a priori estimates in Proposition 5.5 we obtain

$$\begin{aligned} &\tau^2 \int_\Omega \left((v_k(\tau) - r_{\alpha,\beta})_x\right)^2 \mathrm{d}x + \int_{Q_{0,T}} t^2 \left[\partial_t\left(\mathbf{a}(u_k, u_{kx}\right)\right]^2 \mathrm{d}x\mathrm{d}t \\ &\quad \le 5D_{2,T}\left\{1 + \int_\Omega \{u_{0k}^2 + \varepsilon_k (u_{0kx})^2\}\mathrm{d}x\right\}, \end{aligned}$$

whence the conclusion immediately follows (see also (6.8)). □

6.3. Existence for BV-initial data

Let $\{\varepsilon_k\}$ be any sequence such that $\varepsilon_k \to 0^+$ as $k \to \infty$. Throughout this subsection we shall always assume that $u_0 \in BV(\Omega)$ and $\{u_{0k}\} \subseteq C^1(\overline{\Omega})$ satisfies (2.1). Let u_k be the solution to (4.1)–(4.2) with $\varepsilon = \varepsilon_k$ and initial datum u_{0k}.

Proposition 6.5. *There holds*

$$\|v_k - \boldsymbol{a}(u_k, u_{kx})\|_{L^2(Q_{0,T})} = \left\|\sqrt{\varepsilon_k}\, u_{kx} + \varepsilon_k\, u_{kxt}\right\|_{L^2(Q_{0,T})} \to 0\,, \tag{6.48}$$

$$\int_{Q_{0,T}} \varepsilon_k |u_{kxt}\, u_{kx}|\,\mathrm{d}x\mathrm{d}t \to 0\,. \tag{6.49}$$

Proof. The claims immediately follow from the a priori estimate (5.39). □

Proposition 6.6. *Let u and $\{u_k\}$ be the limiting function and the sequence given by Propositions 6.2, 6.3, respectively. Then,*

$$u_t \in L^2(Q_{0,T})\,, \quad u \in L^\infty_w(0,T; BV_{\mathrm{loc}}(\Omega)), \quad h(u, Du) \in L^\infty_{w^*}(0,T; \mathcal{M}_{\mathrm{loc}}(\Omega))\,, \tag{6.50}$$

$$\boldsymbol{a}(u, u_x) \in L^2(0,T; H^1(\Omega))\,, \quad T_{a,b}(u) \in L^1_w(0,T; BV(\Omega)) \tag{6.51}$$

for every $a, b \in \mathbb{R}$, $a < b$. Moreover,

$$u_{kt} \rightharpoonup u_t \quad in \ L^2(Q_{0,T}), \tag{6.52}$$

$$v_k \rightharpoonup \boldsymbol{a}(u, u_x) \quad in \ L^2(0, T; H^1(\Omega)), \tag{6.53}$$

and, for every $\zeta \in C^1([0, T]; C^1_c(\Omega))$, there holds

$$\lim_{k\to\infty} \int_{Q_{0,T}} \boldsymbol{a}(u_k, u_{kx})\, u_{kx}\, \zeta \,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega \zeta(t)\, dh(u(t), Du(t)) \right) \mathrm{d}t\,, \tag{6.54}$$

$$\lim_{k\to\infty} \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left(u_{kx}\right)^2 \zeta \,\mathrm{d}x\mathrm{d}t = 0\,. \tag{6.55}$$

Remark 6.3.

(i) From Proposition 6.6, it follows that $|Du| \in L^\infty_{w*}(0, T; \mathcal{M}_{\mathrm{loc}}(\Omega))$ and, by (5.44) (satisfied by every u_k), (2.1), (6.17) and the Dominated Convergence theorem we get

$$\int_{Q_{0,T}} \zeta\, u_{kx} \,\mathrm{d}x\mathrm{d}t \to \int_0^T \left(\int_\Omega \zeta(t)\, dDu(t) \right) \mathrm{d}t \quad \text{for all } \zeta \in C([0, T]; C_c(\Omega))\,. \tag{6.56}$$

(ii) By (5.44), for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ independent of k such that

$$\|u_{kx}\|_{L^\infty(0,T;L^1(\Omega'))} \le C_{\Omega'}\,, \tag{6.57}$$

$$\|u_k\|_{L^\infty(Q'_{0,T})} \le C_{\Omega'} \tag{6.58}$$

(see also (5.8)) whence, arguing as in Remark 6.1,

$$u_k \to u \quad \text{in } L^p(Q'_{0,T}) \quad \text{for all } 1 \le p < \infty \tag{6.59}$$

and, for $a.e.\ t \in (0, T)$,

$$u_k(t) \to u(t) \quad \text{in } L^p(\Omega') \quad \text{for all } 1 \le p < \infty\,. \tag{6.60}$$

(see also (6.25)). Let us also explicitly observe that the limiting function u belongs to $L^\infty(Q'_{0,T})$ for every Ω' as above.

Proof of Proposition 6.6. Claims in (6.50)–(6.51), and (6.52)–(6.53) follow from (2.1), the a priori estimates (5.39), (5.40), (5.44), (5.46) and the very definition of the measure $h(u, Du)$ in (2.3).

Let us address (6.54)–(6.55). For every nonnegative $\eta \in C^1([0, T])$ and $\rho \in C^1_c(\Omega)$, $\rho \ge 0$, by (4.9) we get

$$\int_{Q_{0,T}} h(u_k, u_{kx})\, \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t + \varepsilon_k \int_{Q_{0,T}} u_{kx}\, u_{kxt}\, \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t$$
$$+ \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left(u_{kx}\right)^2 \eta(t)\, \rho(x) \,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} v_k u_k\, \rho'(x)\, \eta(t) \,\mathrm{d}x\mathrm{d}t$$

$$= \int_{Q_{0,T}} \left\{ \frac{u_k^2}{2}\,\eta'(t)\,\rho(x) + F\,u_k\,\eta(t)\,\rho(x) \right\} \mathrm{d}x\mathrm{d}t$$
$$- \int_{\Omega} \left\{ \frac{u_k^2(T)}{2}\,\eta(T) - \frac{u_{0k}^2}{2}\,\eta(0) \right\} \rho(x)\,\mathrm{d}x\,.$$

By (6.49), (6.53), (6.59) and (6.60) (with $\Omega' \subset\subset \Omega$ such that $\operatorname{supp}\rho \subseteq \Omega'$), letting $k \to \infty$ in the previous equality gives

$$\liminf_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t$$
$$\le \limsup_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t$$
$$\le \limsup_{k\to\infty} \left\{ \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t + \sqrt{\varepsilon_k} \int_{Q_{0,T}} \left(u_{kx}\right)^2 \eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t \right\}$$
$$\le \int_{Q_{0,T}} \left\{ \frac{u^2}{2}\,\eta'(t)\,\rho(x) + F\,u\,\eta(t)\,\rho(x) \right\} \mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_{0,T}} \mathbf{a}(u, u_x)\,u\,\rho'(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t$$
$$- \int_{\Omega} \frac{u^2(T)}{2}\,\eta(T)\,\rho(x)\,\mathrm{d}x + \int_{\Omega} \frac{u_0^2}{2}\,\eta(T)\,\rho(x)\,\mathrm{d}x$$
$$= \int_0^T \left(\int_{\Omega} \mathbf{a}(u(t), u_x(t))\,\rho\,dDu(t) \right) \eta(t)\,\mathrm{d}t$$
$$= \int_0^T \left(\int_{\Omega} \rho(x)\,dh(u(t), Du(t)) \right) \eta(t)\,\mathrm{d}t$$

(see (6.34) and (6.40)). Since arguing as in Proposition 6.4 we have

$$\liminf_{k\to\infty} \int_{Q_{0,T}} h(u_k, u_{kx})\,\eta(t)\,\rho(x)\,\mathrm{d}x\mathrm{d}t \ge \int_0^T \left(\int_{\Omega} \rho(x)\,dh(u(t), Du(t)) \right) \eta(t)\,\mathrm{d}t\,,$$

the conclusion follows from the above inequalities, as in the proof of Proposition 6.4-(iii). □

Proof of Theorem 3.4. Claim (i) follows from Proposition 6.6, whereas claim (ii) is a direct consequence of the a priori estimate (5.45) and (2.1). The proof of equality (3.1) relies on (6.53) and (6.56), letting $k \to \infty$ in the weak formulation of the equality $u_{kxt} = v_{kxx} + F_x$, namely $\int_{Q_{0,T}} u_{kx}\,\zeta_t\,\mathrm{d}x\mathrm{d}t - \int_{Q_{0,T}} v_{kx}\zeta_x\,\mathrm{d}x\mathrm{d}t + \int_{Q_{0,T}} F_x\,\zeta\,\mathrm{d}x\mathrm{d}t = -\int_{\Omega} u_{0kx}\,\zeta(0)\,\mathrm{d}x$ for all $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta(T) = 0$ in Ω. □

7. Monotonicity properties of $D^s_{\pm}u(t)$: proofs

We begin by proving the convergence $a.e.$ in $Q_{0,T}$ of the sequence $\{u_{kx}\}$ to the density u_x of the absolutely continuous part of the measure Du.

Proposition 7.1. *Let* $u_0 \in L^2(\Omega)$ *and* (H_3) *be satisfied. Let* u *and* $\{u_k\}$ *be the strong solution to* $(P^{u_0}_{\alpha,\beta})$ *and the sequence given by Propositions 6.2–6.3, respectively. Then, possibly up to a subsequence of* $\{u_k\}$ *not relabeled, there holds*

$$u_{kx} \to u_x \quad a.e. \ in \ Q_{0,T} \, . \tag{7.1}$$

Remark 7.1. Under the assumptions of Proposition 7.1, by (6.11), (6.13), (6.24) and (7.1), possibly up to a subsequence (not relabeled) there also holds

$$\mathbf{a}(u_k, u_{kx}) \to \mathbf{a}(u, u_x) \quad \text{a.e. in } Q_{0,T} \, , \tag{7.2}$$

$$v_k \to \mathbf{a}(u, u_x) \quad \text{a.e. in } Q_{0,T} \, , \tag{7.3}$$

where $\{v_k\}$ is the sequence in (6.9).

Proof of Proposition 7.1. The claim will follow by a diagonal argument if we prove that, for every $\Omega' \subset\subset \Omega$ and $\tau \in (0,T)$, possibly up to a subsequence (not relabeled), there holds

$$u_{kx} \to u_x \quad a.e. \text{ in } Q'_{\tau,T} \, .$$

To this purpose, let $\Omega' \subset\subset \Omega$, $\tau \in (0,T)$, $\rho \in C^1_c(\Omega')$ and $\eta \in C^1_c((\tau,T))$ be fixed. Then

$$\begin{aligned} &\int_{Q'_{\tau,T}} [\mathbf{a}(u,u_{kx}) - \mathbf{a}(u,u_x)]\,[u_{kx}-u_x]\,\rho(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t \\ &\quad = \int_{Q'_{\tau,T}} [\mathbf{a}(u,u_{kx}) - \mathbf{a}(u_k,u_{kx})]\,[u_{kx}-u_x]\,\rho(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{Q'_{\tau,T}} [\mathbf{a}(u_k,u_{kx}) - \mathbf{a}(u,u_x)]\,[u_{kx}-u_x]\,\rho(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t \, . \end{aligned} \tag{7.4}$$

Let us address the first term in the right-hand side of (7.4). By (6.20) there exists $C_{\Omega',\tau} > 0$ such that $\|u\|_{L^\infty(Q'_{\tau,T})} \le C_{\Omega',\tau}$ and $\|u_k\|_{L^\infty(Q'_{\tau,T})} \le C_{\Omega',\tau}$, whence by (1.9) (with $M = C_{\Omega',\tau}$) and (6.18) there holds

$$\begin{aligned} &\int_{Q'_{\tau,T}} |\mathbf{a}(u,u_{kx}) - \mathbf{a}(u_k,u_{kx})|\,|u_{kx}|\,\mathrm{d}x\mathrm{d}t \\ &\quad \le D_M \int_{Q'_{\tau,T}} |u-u_k|\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } k \to \infty \, . \end{aligned} \tag{7.5}$$

Concerning the second term in the right-hand side of (7.4), by (6.41) we have

$$\lim_{k\to\infty} \int_{Q'_{\tau,T}} \mathbf{a}(u_k,u_{kx})\,u_{kx}\,\rho(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t = \int_0^T \eta(t) \left(\int_\Omega \rho\, dh(u(t), Du(t)) \right) \mathrm{d}t \, , \tag{7.6}$$

whereas from (6.17), (6.19) and the Dominated Convergence theorem we get

$$\lim_{k\to\infty} \int_{Q'_{\tau,T}} \mathbf{a}(u,u_x)\,u_{kx}\,\rho(x)\,\eta(t)\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_\tau^T \eta(t) \lim_{k\to\infty} \left(\int_\Omega \mathbf{a}(u(t), u_x(t))\, u_{kx}\, \rho(x)\, \mathrm{d}x \right) \mathrm{d}t \\
&= \int_0^T \eta(t) \left(\int_\Omega \mathbf{a}(u(t), u_x(t))\, \rho(x)\, dDu(t) \right) \mathrm{d}t \\
&= \int_0^T \eta(t) \left(\int_\Omega \rho\, dh(u(t), Du(t)) \right) \mathrm{d}t\,,
\end{aligned}
$$

the latter equality following by (6.40) (here we have also used that $\rho\, \mathbf{a}(u(t), u_x(t)) \in C_c(\Omega)$ for $a.e.\ t \in (\tau, T)$). Combining the above equality and (7.6) gives

$$
\lim_{k\to\infty} \int_{Q'_{\tau,T}} [\mathbf{a}(u_k, u_{kx}) - \mathbf{a}(u, u_x)]\ u_{kx}\, \rho(x)\, \eta(t)\, \mathrm{d}x\mathrm{d}t = 0\,. \tag{7.7}
$$

In view of (6.32), we also have

$$
\lim_{k\to\infty} \int_{Q'_{\tau,T}} [\mathbf{a}(u_k, u_{kx}) - \mathbf{a}(u, u_x)]\ u_x\, \rho(x)\, \eta(t)\, \mathrm{d}x\mathrm{d}t = 0\,. \tag{7.8}
$$

To sum up, by (7.5), (7.7) and (7.8), letting $k \to \infty$ in (7.4) we get

$$
\lim_{k\to\infty} \int_{Q'_{\tau,T}} [\mathbf{a}(u, u_{kx}) - \mathbf{a}(u, u_x)]\ [u_{kx} - u_x]\ \rho(x)\, \eta(t)\, \mathrm{d}x\mathrm{d}t = 0
$$

for every $\rho \in C_c^1(\Omega')$ and $\eta \in C_c^1((\tau, T))$. Since $[\mathbf{a}(u, u_{kx}) - \mathbf{a}(u, u_x)]\ [u_{kx} - u_x] \geq 0$ $a.e.$ in $Q_{0,T}$, this proves that possibly up to a subsequence (not relabeled)

$$
[\mathbf{a}(u(x,t), u_{kx}(x,t)) - \mathbf{a}(u(x,t), u_x(x,t))]\ [u_{kx}(x,t) - u_x(x,t)] \overset{k\to\infty}{\longrightarrow} 0 \tag{7.9}
$$

for $a.e.\ (x, t) \in Q'_{\tau,T}$. Therefore the conclusion follows from this convergence, using the strictly increasing character of the map $\xi \to \mathbf{a}(z, \xi)$ (see assumption (H_3)). □

Proposition 7.2. *Let (H_3) be satisfied. Let u and $\{u_k\}$ be the strong solution to $(P^{u_0}_{\alpha,\beta})$ with initial datum $u_0 \in BV(\Omega)$ and the sequence given by Propositions 6.6 and 7.1, respectively. Then, for every $\Omega' \subset\subset \Omega$ there exist $\lambda_1, \lambda_2 \in L^\infty_{w*}(0, T; \mathcal{M}^+(\Omega'))$ such that*

$$
\lambda_1(t) - \lambda_2(t) = D^s u(t) \quad \text{in } \mathcal{M}(\Omega') \quad \text{for } a.e.\ t \in (0, T) \tag{7.10}
$$

and, possibly up to a subsequence of $\{u_k\}$ not relabeled, for all $\psi \in C_c(Q'_{0,T})$ there holds

$$
\lim_{k\to\infty} \int_{Q'_{0,T}} [u_{kx}]^+\, \psi\, \mathrm{d}x\mathrm{d}t = \int_{Q'_{0,T}} \psi\ [u_x]^+\, \mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \psi(t)\, d\lambda_1(t) \right) \mathrm{d}t\,, \tag{7.11}
$$

$$
\lim_{k\to\infty} \int_{Q'_{0,T}} [u_{kx}]^-\, \psi\, \mathrm{d}x\mathrm{d}t = \int_{Q'_{0,T}} \psi\ [u_x]^-\, \mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \psi(t)\, d\lambda_2(t) \right) \mathrm{d}t\,. \tag{7.12}
$$

Proof. Let $\Omega' \subset\subset \Omega$ be fixed and let $\{\tau_k^\pm\} \subseteq \mathcal{Y}(Q'_{0,T};\mathbb{R})$ be the sequences of Young measures associated with $\left\{[u_{kx}]^\pm\right\}$ ([26]). By (5.44) and the Prokhorov's theorem, there exist $\tau^\pm \in \mathcal{Y}(Q'_{0,T};\mathbb{R})$ and a subsequence of $\{u_k\}$ (not relabeled) such that

$$\tau_k^\pm \to \tau^\pm \text{ narrowly in } Q'_{0,T} \times \mathbb{R}$$

(see [7]). In view of (7.1), the disintegrations $\tau^\pm_{(x,t)}$ of $\tau^\pm$ satisfy

$$\tau^+_{(x,t)} = \delta_{\{[u_x(x,t)]^+\}}\,, \quad \tau^-_{(x,t)} = \delta_{\{[u_x(x,t)]^-\}} \quad \text{for } a.e.\ (x,t) \in Q \tag{7.13}$$

(*e.g.*, see [26]). Let us only address (7.11), the proof of (7.12) being analogous. By the boundedness of $\{u_{kx}\}$ in $L^\infty(0,T;L^1(\Omega'))$ (see (5.44)) and the Biting lemma (*e.g.*, see [26]), there exists a sequence of measurable sets $A_j \subseteq Q'_{0,T}$, $A_j \subseteq A_{j+1}$ with Lebesgue measure $|A_j| \to 0$, such that, possibly up to a subsequence $\{[u_{k_jx}]^+\} \subseteq \{[u_{kx}]^+\}$, there holds

$$[u_{k_jx}]^+ \chi_{Q'_{0,T}\setminus A_j} \rightharpoonup \int_{\mathbb{R}} \xi d\tau^+_{(\cdot,\cdot)} = [u_x]^+ \text{ in } L^1\big(Q'_{0,T}\big) \tag{7.14}$$

(see (7.13)). By (5.44) the sequence $\left\{[u_{k_jx}]^+\chi_{A_j}\right\}$ is bounded in $L^\infty(0,T;L^1(\Omega'))$, hence there exists a Radon measure $\lambda_1 \in L^\infty(0,T;\mathcal{M}^+(\Omega'))$ such that (possibly up to a subsequence)

$$[u_{k_jx}]^+\chi_{A_j} \overset{*}{\rightharpoonup} \lambda_1 \text{ in } \mathcal{M}(Q'_{0,T})\,. \tag{7.15}$$

Thus, by (7.14) and (7.15) we obtain (7.11).

Finally, combining (7.11)–(7.12) with (6.56) plainly gives, for $a.e.\ t \in (0,T)$,

$$\begin{aligned} Du(t) &= [u_x(t)]^+\mathcal{L} - [u_x(t)]^-\mathcal{L} + \lambda_1(t) - \lambda_2(t) \\ &= u_x(t)\,\mathcal{L} + \lambda_1(t) - \lambda_2(t) \quad \text{in } \mathcal{M}(\Omega')\,, \end{aligned}$$

whence equality (7.10) follows at once by the uniqueness of the Lebesgue decomposition of the measure $Du(t)$. □

For every $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $g' \geq 0$ in $\mathbb{R}$, satisfying either

$$g(y) = -1 \text{ for all } y \leq 0 \text{ and } \exists\, d \in (0,1) \text{ such that } g(y) = 0 \text{ for all } y \geq 1-d, \tag{7.16}$$

or

$$g(y) = 1 \text{ for all } y \geq 0 \text{ and } \exists\, d \in (0,1) \text{ such that } g(y) = 0 \text{ for all } y \leq -1+d, \tag{7.17}$$

set, for every $(z,\xi) \in \mathbb{R}^2$,

$$G(z,\xi) := \int_0^\xi g(\mathbf{a}(z,s))\,\mathrm{d}s\,. \tag{7.18}$$

Lemma 7.3. *Let* (H_3) *be satisfied and, for every* $u_0 \in BV(\Omega)$*, let* u *be the strong solution to* $(P^{u_0}_{\alpha,\beta})$*. Then, for every* $\Omega' \subset\subset \Omega$ *and* $\zeta \in C^1([0,T]; C^1_c(\Omega'))$*,* $\zeta \geq 0$*,* $\zeta(T) = 0$ *and* $\zeta_t \in C_c(Q'_{0,T})$*, there holds:*

(i) *for all* $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$*,* $g' \geq 0$ *in* $\mathbb{R}$*, satisfying* (7.16)*,*

$$\begin{aligned}
&\int_{Q'_{0,T}} G(u, u_x)\, \zeta_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_2(t) \right) \mathrm{d}t \\
&\quad + \int_{Q'_{0,T}} H_g\, \zeta \,\mathrm{d}x\mathrm{d}t + \int_{\Omega'} G(u_0, u_{0x})\, \zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\, dD^s_- u_0 \\
&\quad - \int_{Q'_{0,T}} g(\boldsymbol{a}(u,u_x))\, \zeta_x\, [\boldsymbol{a}(u,u_x)]_x \,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{Q'_{0,T}} \left\{ g'(\boldsymbol{a}(u,u_x))\; ([\boldsymbol{a}(u,u_x)]_x)^2\, \zeta - g(\boldsymbol{a}(u,u_x))\, F_x\, \zeta \right\} \mathrm{d}x\mathrm{d}t\,; \qquad (7.19)
\end{aligned}$$

(ii) *for all* $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$*,* $g' \geq 0$ *in* $\mathbb{R}$*, satisfying* (7.17)*,*

$$\begin{aligned}
&\int_{Q'_{0,T}} G(u, u_x)\, \zeta_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_1(t) \right) \mathrm{d}t \\
&\quad + \int_{Q'_{0,T}} H_g\, \zeta \,\mathrm{d}x\mathrm{d}t + \int_{\Omega'} G(u_0, u_{0x})\, \zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\, dD^s_+ u_0 \\
&\quad - \int_{Q'_{0,T}} g(\boldsymbol{a}(u,u_x))\, \zeta_x\, [\boldsymbol{a}(u,u_x)]_x \,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{Q'_{0,T}} \left\{ g'(\boldsymbol{a}(u,u_x))\; ([\boldsymbol{a}(u,u_x)]_x)^2\, \zeta - g(\boldsymbol{a}(u,u_x))\, F_x\, \zeta \right\} \mathrm{d}x\mathrm{d}t\,. \qquad (7.20)
\end{aligned}$$

Here, λ_1 *and* λ_2 *are the measures given in Proposition* 7.2 *in correspondence with* Ω'*,* G *is the function in* (7.18) *and*

$$\begin{aligned}
&H_g(x,t) \\
&:= u_t(x,t) \int_0^{u_x(x,t)} \left[g'\big(\mathbf{a}(u(x,t),s)\big)\, \partial_z \mathbf{a}(u(x,t),s) \right] \mathrm{d}s \quad (a.e.\ (x,t) \in Q_{0,T}). \qquad (7.21)
\end{aligned}$$

In view of (7.16)–(7.17), it will be easily seen in the proof of Lemma 7.3 that $H_g \in L^2(Q'_{0,T})$ for all $\Omega' \subset\subset \Omega$ (see (7.30), (7.32) and (7.33) below).

Proof. We shall only address claim (i), the proof of (ii) being analogous.

For every $u_0 \in BV(\Omega)$, let u_{0k} ($\varepsilon_k \to 0^+$ as $k \to \infty$) satisfy (2.1). Fix any $\Omega' \subset\subset \Omega$ and $\zeta \in C^1([0,T]; C^1_c(\Omega'))$, $\zeta \geq 0$, $\zeta(T) = 0$ and $\zeta_t \in C_c(Q'_{0,T})$. Let $u_k \equiv u_{\varepsilon_k}$ be the subsequence given in Proposition 7.2. For every such ζ and for every nondecreasing $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ as in (7.16), the conclusion will follow letting $\varepsilon \equiv \varepsilon_k \to 0$ in inequalities (4.17). To this aim, we begin by observing that the sequence $\{u_k\}$ is bounded in $L^\infty(Q'_{0,T})$, namely

$$\|u_k\|_{L^\infty(Q'_{0,T})} \leq M \qquad (7.22)$$

for some $M > 0$ depending on Ω' (see (6.58)). In addition, since g and g' are continuous and bounded, by (7.3) and the Dominated Convergence theorem there holds

$$g'(v_k) \to g'(\mathbf{a}(u, u_x)), \quad g(v_k) \to g(\mathbf{a}(u, u_x)) \;\; a.e. \text{ in } Q_{0,T} \text{ and in } L^p(Q_{0,T}), \tag{7.23}$$

for any $1 \le p < \infty$. Combining the above convergence and (6.53) plainly gives

$$\begin{aligned} &\int_{Q'_{0,T}} g(\mathbf{a}(u, u_x))\, \zeta_x\, [\mathbf{a}(u, u_x)]_x \,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{Q'_{0,T}} \left\{ g'(\mathbf{a}(u, u_x))\, ([\mathbf{a}(u, u_x)]_x)^2\, \zeta - g(\mathbf{a}(u, u_x))\, F_x\, \zeta \right\} \mathrm{d}x\mathrm{d}t \\ &\le \liminf_{k\to\infty} \int_{Q'_{0,T}} \left\{ g(v_k)\, \zeta_x\, v_{kx} + g'(v_k)\, v_{kx}^2\, \zeta - g(v_k)\, F_x\, \zeta \right\} \mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{7.24}$$

In order to address the remaining terms in inequalities (4.17), we observe that:

- for every $z \in \mathbb{R}$ the map $s \mapsto \mathbf{a}(z, s) + \sqrt{\varepsilon_k} s$ is strictly increasing (see assumption (H_3)) and $\mathbf{a}(z, 0) = 0$, hence by (7.16) we get

  $$G_{\varepsilon_k}(z, \xi) = \int_0^{\xi} g(\mathbf{a}(z, s) + \sqrt{\varepsilon_k} s)\, \mathrm{d}s = -\xi \quad \text{for all } z \in \mathbb{R} \text{ and } \xi \le 0$$

  (here G_{ε_k} is the function in (4.16));
- let $z \in [-M, M]$, with M as in (7.22), $k \in \mathbb{N}$ and $d \in (0, 1)$ be arbitrarily fixed, and let $s_{d,z,k} > 0$ be the solution to the equation

  $$\mathbf{a}(z, s) + \sqrt{\varepsilon_k} s = 1 - d\,.$$

  Then, by the strictly increasing character of the map $s \mapsto \mathbf{a}(z, s)$ it follows that $\mathbf{a}(z, s) + \sqrt{\varepsilon_k} s > 1 - d$ for all $s > s_{d,z,k}$, whereas by assumption (H_3) (in particular, see (1.10)) we have

  $$\begin{aligned} 1 - \frac{C^-_{1,M}}{\varphi_{1,M}(z) + |s_{d,z,k}|^{\sigma_1}} &\le \mathbf{a}(z, s_{d,z,k}) + \sqrt{\varepsilon_k} s_{d,z,k} \\ &= 1 - d \;\Rightarrow\; s_{d,z,k} \le \left(\frac{C^-_{1,M}}{d} \right)^{1/\sigma_1} =: \gamma_{d,M} \end{aligned} \tag{7.25}$$

  (here we have also used that $\varphi_{1,M}(z) > 0$ for every z; see assumption (H_3)). Notice that the estimate in the right-hand side of (7.25) is independent of $z \in [-M, M]$ and $k \in \mathbb{N}$.

By the above considerations and (7.16), it follows that

$$\begin{aligned} G_{\varepsilon_k}(u_k(x,t), u_{kx}(x,t)) &= -\, u_{kx}(x,t) \quad \text{if } u_{kx}(x,t) \le 0\,, \\ G_{\varepsilon_k}(u_k(x,t), u_{kx}(x,t)) &= \int_0^{\gamma_{d,M}} g(\mathbf{a}(u_k(x,t), s) + \sqrt{\varepsilon_k} s)\, \mathrm{d}s \quad \text{if } u_{kx}(x,t) \ge \gamma_{d,M}\,, \end{aligned}$$

whence

$$
\begin{aligned}
& G_{\varepsilon_k}(u_k(x,t), u_{kx}(x,t)) - [u_{kx}(x,t)]^- \\
&= \begin{cases} 0 & \text{if } \ u_{kx}(x,t) \le 0\,, \\ \displaystyle\int_0^{\gamma_{d,M}} g(\mathbf{a}(u_k(x,t),s) + \sqrt{\varepsilon_k} s)\, \mathrm{d}s & \text{if } \ u_{kx}(x,t) \ge \gamma_{d,M}\,. \end{cases}
\end{aligned}
$$

In view of the above equality, since the value $\gamma_{d,M}$ is independent of $k \in \mathbb{N}$ and $g \in L^\infty(\mathbb{R})$, the sequence $\left\{G_{\varepsilon_k}(u_k, u_{kx}) - [u_{kx}]^-\right\}$ is bounded in $L^\infty(Q'_{0,T})$, whence by the Dominated Convergence theorem

$$
G_{\varepsilon_k}(u_k, u_{kx}) - [u_{kx}]^- \to G(u, u_x) - [u_x]^- \quad \text{in } \ L^p(Q'_{0,T}) \tag{7.26}
$$

for every $1 \le p < \infty$ (recall that $u_k \to u$ and $u_{kx} \to u_x$ $a.e.$ in $Q'_{0,T}$; see (6.24) and (7.1)). Since $\zeta_t \in C_c(Q'_{0,T})$, combining (7.26) and (7.12) plainly gives

$$
\begin{aligned}
& \lim_{k\to\infty} \int_{Q'_{0,T}} G_{\varepsilon_k}(u_k, u_{kx}) \zeta_t \mathrm{d}x\mathrm{d}t \\
& \quad = \lim_{k\to\infty} \int_{Q'_{0,T}} \left\{G_{\varepsilon_k}(u_k, u_{kx}) - [u_{kx}]^-\right\} \zeta_t \mathrm{d}x\mathrm{d}t + \lim_{k\to\infty} \int_{Q'_{0,T}} [u_{kx}]^- \zeta_t \mathrm{d}x\mathrm{d}t \\
& \quad = \int_{Q'_{0,T}} G(u, u_x)\zeta_t \, \mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_\Omega \zeta_t(t) d\lambda_2(t)\right) \mathrm{d}t.
\end{aligned} \tag{7.27}
$$

Analogously, it can be checked that the sequence $\left\{G_{\varepsilon_k}(u_{0k}, u_{0kx}) - [u_{0kx}]^-\right\}$ is bounded in $L^\infty(\Omega')$, hence by (2.1) we have

$$
G_{\varepsilon_k}(u_{0k}, u_{0kx}) - [u_{0kx}]^- \to G(u_0, u_{0x}) - [u_{0x}]^- \quad \text{in } \ L^p(\Omega') \qquad (1 \le p < \infty)\,,
$$

$$
\begin{aligned}
& \lim_{k\to\infty} \int_{\Omega'} G_{\varepsilon_k}(u_{0k}, u_{0kx})\, \zeta(x,0)\, \mathrm{d}x \\
& \quad = \lim_{k\to\infty} \int_{\Omega'} \left(G_{\varepsilon_k}(u_{0k}, u_{0kx}) - [u_{0kx}]^-\right) \zeta(x,0)\, \mathrm{d}x + \lim_{k\to\infty} \int_{\Omega'} [u_{0kx}]^- \zeta(x,0)\, \mathrm{d}x \\
& \quad = \int_{\Omega'} G(u_0, u_{0x})\, \zeta(x,0)\, \mathrm{d}x + \int_{\Omega'} \zeta(x,0)\, dD^s_- u_0\,.
\end{aligned} \tag{7.28}
$$

Then, inequality (7.19) will follow from (7.24), (7.27) and (7.28) letting $\varepsilon_k \to 0$ in (4.17) if we prove that

$$
\lim_{k\to\infty} \int_{Q'_{0,T}} H_{\varepsilon_k, g}\, \zeta \, \mathrm{d}x\mathrm{d}t = \int_{Q'_{0,T}} H_g\, \zeta \, \mathrm{d}x\mathrm{d}t\,, \tag{7.29}
$$

where H_g is the function in (7.21) and

$$
H_{\varepsilon_k, g}(x,t) = u_{kt}(x,t)\, \tilde{H}_{\varepsilon_k, g}(x,t) \qquad ((x,t) \in Q_{0,T})\,, \tag{7.30}
$$

$$\tilde H_{\varepsilon_k,g}(x,t) := \int_0^{u_{kx}(x,t)} \left[g'\big(\mathbf{a}(u_k(x,t),s)+\sqrt{\varepsilon_k}s\big)\partial_z\mathbf{a}(u_k(x,t),s)\right] \mathrm{d}s \quad ((x,t)\in Q_{0,T})$$

(see (4.18)). Since $\mathbf{a}(z,\xi)\le 0$ for all $z\in\mathbb{R}$ and $\xi\le 0$, $g'\ge 0$ in $\mathbb{R}_+$ and $g'=0$ in $\mathbb{R}_-$ and in $[1-d,\infty)$ (see (7.16)), we begin by observing that, for all $(x,t)\in Q'_{0,T}$, we have

$$\tilde H_{\varepsilon_k,g}(x,t) = 0 \quad \text{if} \quad u_{kx}(x,t)\le 0\,, \tag{7.31}$$

and (see (7.25))

$$\begin{aligned}\left|\tilde H_{\varepsilon_k,g}(x,t)\right| &\le \|\partial_z\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \int_0^{[u_{kx}(x,t)]^+} g'\big(\mathbf{a}(u_k(x,t),s)+\sqrt{\varepsilon_k}s\big)\,\mathrm{d}s\\ &\le \|\partial_z\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \int_0^{\gamma_{d,M}} g'(\mathbf{a}(u_k(x,t),s)+\sqrt{\varepsilon_k}s)\mathrm{d}s \text{ if } u_{kx}(x,t)\ge 0 \end{aligned}\tag{7.32}$$

(recall that $\mathbf{a}(\cdot,\cdot)\in \mathrm{Lip}(\mathbb{R}^2)$). From (7.31) and (7.32) we obtain that the sequence $\{\tilde H_{\varepsilon_k,g}\}$ is bounded in $L^\infty(Q'_{0,T})$; moreover, as $\partial_z\mathbf{a}(\cdot,\cdot)\in C(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ (see assumptions (H_1) and (H_3)) and $g\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, the convergences in (6.24) and (7.1) give

$$\tilde H_{\varepsilon_k,g}(x,t)\to \int_0^{u_x(x,t)} \left[g'\big(\mathbf{a}(u(x,t),s)\big)\partial_z\mathbf{a}(u(x,t),s)\right]\mathrm{d}s \text{ for } a.e.\ (x,t)\in Q'_{0,T}, \tag{7.33}$$

whence by the Dominated Convergence theorem it follows that

$$\tilde H_{\varepsilon_k,g}\to \int_0^{u_x}\left[g'\big(\mathbf{a}(u,s)\big)\,\partial_z\mathbf{a}(u,s)\right]\mathrm{d}s \text{ in } L^p(Q'_{0,T}) \text{ for all } 1\le p<\infty.$$

From this convergence and (6.52) we get $H_{\varepsilon_k,g}\rightharpoonup H_g$ in $L^1(Q'_{0,T})$, and the conclusion follows. □

For every $\delta\in(0,1)$, let $\overline{g}_\delta,\underline{g}_\delta:\mathbb{R}\to\mathbb{R}$ be defined by

$$\overline{g}_\delta(y) = -1\chi_{]-\infty,1]} + \frac{1}{\delta}(y-1+\delta)\chi_{\{1-\delta\le y\le 1\}} \tag{7.34}$$

and

$$\underline{g}_\delta(y) = 1\chi_{[-1,\infty[} + \frac{1}{\delta}(y+1-\delta)\chi_{\{-1\le y\le -1+\delta\}}\,, \tag{7.35}$$

and set

$$\underline{\overline{G}}_\delta(z,\xi) := \int_0^\xi \underline{g}_\delta(\mathbf{a}(z,s))\,\mathrm{d}s \qquad ((z,\xi)\in\mathbb{R}^2)\,. \tag{7.36}$$

Lemma 7.4. *Let (H_3) be satisfied and, for every $u_0\in BV(\Omega)$, let u be the strong solution to $(P^{u_0}_{\alpha,\beta})$. Then, for every $\Omega'\subset\subset\Omega$ and $\zeta\in C^1([0,T];C^1_c(\Omega'))$, $\zeta\ge 0$, $\zeta(T)=0$ and $\zeta_t\in C_c(Q'_{0,T})$, there holds:*

(i) *for every* $\delta \in (0,1)$,

$$\begin{aligned}
&\int_{Q'_{0,T}} \underline{G}_\delta(u,u_x)\,\zeta_t\,\mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_2(t)\right)\mathrm{d}t + \int_{Q'_{0,T}} H_{\underline{g}_\delta}\,\zeta\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega'} \underline{G}_\delta(u_0,u_{0x})\,\zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\,dD^s_- u_0 \\
&\quad - \int_{Q'_{0,T}} \underline{g}_\delta(\boldsymbol{a}(u,u_x))\,\zeta_x\,[\boldsymbol{a}(u,u_x)]_x\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{Q'_{0,T}} \left\{\frac{1}{\delta}\left([\boldsymbol{a}(u,u_x)]_x\right)^2 \chi_{\{\boldsymbol{a}(u,u_x)>1-\delta\}}\,\zeta - \underline{g}_\delta(\boldsymbol{a}(u,u_x))\,F_x\,\zeta\right\}\mathrm{d}x\mathrm{d}t\,;
\end{aligned} \tag{7.37}$$

(ii) *for every* $\delta \in (0,1)$,

$$\begin{aligned}
&\int_{Q'_{0,T}} \overline{G}_\delta(u,u_x)\,\zeta_t\,\mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_1(t)\right)\mathrm{d}t + \int_{Q'_{0,T}} H_{\overline{g}_\delta}\,\zeta\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega'} \overline{G}_\delta(u_0,u_{0x})\,\zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\,dD^s_+ u_0 \\
&\quad - \int_{Q'_{0,T}} \overline{g}_\delta(\boldsymbol{a}(u,u_x))\,\zeta_x\,[\boldsymbol{a}(u,u_x)]_x\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{Q'_{0,T}} \left\{\frac{1}{\delta}\left([\boldsymbol{a}(u,u_x)]_x\right)^2 \chi_{\{\boldsymbol{a}(u,u_x)<-1+\delta\}}\,\zeta - \overline{g}_\delta(\boldsymbol{a}(u,u_x))\,F_x\,\zeta\right\}\mathrm{d}x\mathrm{d}t\,.
\end{aligned} \tag{7.38}$$

Here λ_1 *and* λ_2 *are the measures obtained in Proposition* 7.2 *in correspondence with* Ω' *and*

$$\begin{cases}
H_{\underline{g}_\delta}(x,t) := \chi_{\{\mathbf{a}(u,u_x)\geq 1-\delta\}} u_t(x,t) \displaystyle\int_0^{u_x(x,t)} \underline{g}'_\delta(\mathbf{a}(u(x,t),s))\partial_z \mathbf{a}(u(x,t),s)\,\mathrm{d}s \\
H_{\overline{g}_\delta}(x,t) := \chi_{\{\mathbf{a}(u,u_x)\leq -1+\delta\}} u_t(x,t) \displaystyle\int_0^{u_x(x,t)} \overline{g}'_\delta(\mathbf{a}(u(x,t),s))\partial_z \mathbf{a}(u(x,t),s)\,\mathrm{d}s\,,
\end{cases} \tag{7.39}$$

for a.e. $(x,t) \in Q'_{0,T}$.

It will be seen below (see (7.43) and (7.44)) that, in view of assumption (H_3), for every $\Omega' \subset\subset \Omega$ the function $H_{g,\delta}$ in (7.39) belongs to the space $L^1(\Omega' \times (0,T))$.

Proof. We shall only prove (7.37), the proof of (7.38) being analogous. To this aim, for every $\delta \in (0,1)$ and $j \in \mathbb{N}$ such that $0 < 1-\delta-3/j < 1-\delta < 1-2/j$, let us set

$$\underline{g}_{\delta,j}(y) = -1\chi_{]-\infty,1]}(y) + \int_0^y \underline{g}'_{\delta,j}(p)\,\mathrm{d}p\,,$$

where

$$\underline{g}'_{\delta,j}(y) = \frac{j}{3\delta}(y - 1 + \delta + 3/j)\chi_{\{1-\delta-3/j<y<1-\delta\}} + \frac{1}{\delta}\chi_{\{1-\delta\le y\le 1-2/j\}} + \frac{j}{\delta}(1 - 1/j - y)\chi_{\{1-2/j<y<1-1/j\}}\,.$$

Notice that $\underline{g}_{\delta,j}$ satisfies (7.16) with $d = 1 - \frac{1}{j}$, $0 \le \underline{g}'_{\delta,j}(y) \le 1/\delta$ for all $y \in \mathbb{R}$ and, as $j \to \infty$,

$$\underline{g}'_{\delta,j}(y) \to \begin{cases} 0 & \text{for } y < 1-\delta \text{ and } y \ge 1 \\ 1/\delta & \text{for } 1-\delta \le y < 1\,, \end{cases} \tag{7.40}$$

$$\underline{g}_{\delta,j}(y) \to \underline{g}_{\delta}(y) \quad \text{for every } \; y \in \mathbb{R}\,. \tag{7.41}$$

Then claim (i) will follow if we take the limit as $j \to \infty$ in inequality (7.19), written for $g = \underline{g}_{\delta,j}$ and

$$G(z,\xi) = \underline{G}_{\delta,j}(z,\xi) := \int_0^{\xi} \underline{g}_{\delta,j}(\mathbf{a}(z,s))\,\mathrm{d}s \qquad ((z,\xi)\in\mathbb{R}^2)\,.$$

Let us fix any $\Omega' \subset\subset \Omega$ and $\zeta \in C^1([0,T]; C^1_c(\Omega'))$, $\zeta \ge 0$, $\zeta(T) = 0$ and $\zeta_t \in C_c(Q'_{0,T})$. Since for $a.e.$ $(x,t) \in Q'_{0,T}$ there holds

$$\underline{g}_{\delta,j}(\mathbf{a}(u(x,t),s)) \to \underline{g}_{\delta}(\mathbf{a}(u(x,t),s)) \quad \text{for all } s \in \mathbb{R},$$

whence $\underline{G}_{\delta,j}(u(x,t),u_x(x,t)) \to \underline{G}_{\delta}(u(x,t),u_x(x,t))$ as $j \to \infty$, and

$$|\underline{G}_{\delta,j}(u(x,t),u_x(x,t))| \le \|\underline{g}_{\delta,j}\|_{L^\infty(\mathbb{R})}|u_x(x,t)| \le |u_x(x,t)| \in L^1(Q'_{0,T})\,,$$

by the Dominated Convergence theorem we get

$$\lim_{j\to\infty}\int_{Q'_{0,T}} \underline{G}_{\delta,j}(u,u_x)\,\zeta_t\,\mathrm{d}x\mathrm{d}t = \int_{Q'_{0,T}} \underline{G}_{\delta}(u,u_x)\zeta_t\,\mathrm{d}x\mathrm{d}t\,.$$

Analogously, relying on (7.41), it can be checked that

$$\lim_{j\to\infty}\int_{\Omega'} \underline{G}_{\delta,j}(u_0,u_{0x})\,\zeta(x,0)\,\mathrm{d}x = \int_{\Omega'} \underline{G}_{\delta}(u_0,u_{0x})\,\zeta(x,0)\,\mathrm{d}x\,,$$

$$\lim_{j\to\infty}\int_{Q'_{0,T}} \underline{g}_{\delta,j}(\mathbf{a}(u,u_x))\,\{F_x\,\zeta-[\mathbf{a}(u,u_x)]_x\,\zeta_x\}\;\mathrm{d}x\mathrm{d}t$$
$$=\int_{Q'_{0,T}} \underline{g}_{\delta}(\mathbf{a}(u,u_x))\,\{F_x\,\zeta-[\mathbf{a}(u,u_x)]_x\,\zeta_x\}\;\mathrm{d}x\mathrm{d}t\,.$$

For $a.e.\ (x,t)\in Q'_{0,T}$ we have

$$\int_{Q'_{0,T}} \underline{g}'_{\delta,j}(\mathbf{a}(u,u_x))\big([\mathbf{a}(u,u_x)]_x\big)^2\zeta\;\mathrm{d}x\mathrm{d}t$$
$$\geq \frac{1}{\delta}\int_{Q'_{0,T}} \chi_{\{1-\delta<\mathbf{a}(u,u_x)<1-2/j\}}\big([\mathbf{a}(u,u_x)]_x\big)^2\zeta\;\mathrm{d}x\mathrm{d}t\,,$$

whence

$$\frac{1}{\delta}\int_{Q'_{0,T}} \chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\big([\mathbf{a}(u,u_x)]_x\big)^2\zeta\;\mathrm{d}x\mathrm{d}t$$
$$=\frac{1}{\delta}\int_{Q'_{0,T}} \chi_{\{1-\delta<\mathbf{a}(u,u_x)<1\}}\big([\mathbf{a}(u,u_x)]_x\big)^2\zeta\;\mathrm{d}x\mathrm{d}t$$
$$\leq \liminf_{j\to\infty}\int_{Q'_{0,T}} \underline{g}'_{\delta,j}(\mathbf{a}(u,u_x))\big([\mathbf{a}(u,u_x)]_x\big)^2\zeta\;\mathrm{d}x\mathrm{d}t\,;$$

here we have used that $\mathbf{a}(u,u_x)<1$ $a.e.$ in $Q'_{0,T}$, since $u_x\in L^1(Q'_{0,T})$, the map $\xi\mapsto\mathbf{a}(z,\xi)$ is strictly increasing by assumption (H_3), and $\mathbf{a}(z,\xi)\to 1$ as $\xi\to\infty$, for all $z\in\mathbb{R}$. Therefore, it only remains to prove that

$$\lim_{j\to\infty}\int_{Q'_{0,T}} H_{\underline{g}_{\delta,j}}\,\zeta\;\mathrm{d}x\mathrm{d}t=\int_{Q'_{0,T}} H_{\underline{g}_{\delta}}\,\zeta\;\mathrm{d}x\mathrm{d}t\,,\tag{7.42}$$

where

$$H_{\underline{g}_{\delta,j}}(x,t)=u_t(x,t)\int_0^{u_x(x,t)} \underline{g}'_{\delta,j}(\mathbf{a}(u(x,t),s))\partial_z\mathbf{a}(u(x,t),s)\,\mathrm{d}s\;\;(a.e.\ (x,t)\in Q'_{0,T}).$$

In this direction, by the strictly increasing character of the mapping $\xi\to\mathbf{a}(z,\xi)$ and the definition of $\underline{g}'_{\delta,j}$, for $a.e\ (x,t)\in Q'_{0,T}$ we have

$$H_{\underline{g}_{\delta,j}}(x,t)=0\quad\text{if}\ \mathbf{a}(u(x,t),u_x(x,t))<1-\delta-3/j$$

(hence if $u_x(x,t)<0$, since in this case $\mathbf{a}(u(x,t),u_x(x,t))<\mathbf{a}(u(x,t),0)=0$; see (H_1) and (H_3)) and

$$
\begin{aligned}
|H_{\underline{g}_{\delta,j}}(x,t)| \leq & \frac{|u_t(x,t)|}{\delta} \int_0^{[u_x(x,t)]^+} |\partial_z \mathbf{a}(u(x,t),s)| \, \mathrm{d}s \\
\leq & \frac{|u_t(x,t)|}{\delta} \int_0^{[u_x(x,t)]^+} \frac{D_M}{(1+|s|)^{\sigma_0}} \mathrm{d}s \\
\leq & \frac{|u_t(x,t)|}{\delta} \int_0^{[u_x(x,t)]^+} \frac{D_M}{(1+|s|)} \mathrm{d}s \\
\leq & \frac{D_M \, |u_t(x,t)|}{\delta} \log(1+|u_x(x,t)|) \in L^1(Q'_{0,T})
\end{aligned}
\tag{7.43}
$$

(recall that $0 \leq \underline{g}'_{\delta,j} \leq 1/\delta$), where D_M is the constant in (H_3) associated to some $M \geq \|u\|_{L^\infty(Q'_{0,T})}$ (see Remark 6.3). Moreover, in view of (7.40) and since for every $z \in \mathbb{R}$ and $s \in \mathbb{R}$, $\mathbf{a}(z,s) < \lim_{s\to+\infty} \mathbf{a}(z,s) = 1$, for $a.e.$ $(x,t) \in Q'_{0,T}$ there holds

$$
\underline{g}'_{\delta,j}(\mathbf{a}(u(x,t),s)) \to \begin{cases} 0 \ \text{ if } \ \mathbf{a}(u(x,t),s) < 1-\delta \\ \\ \frac{1}{\delta} \ \text{ if } \ \mathbf{a}(u(x,t),s) \geq 1-\delta \end{cases} \quad \text{as } j \to \infty\,,
$$

whence $\underline{g}'_{\delta,j}(\mathbf{a}(u(x,t),s)) \to \underline{g}'_{\delta}(\mathbf{a}(u(x,t),s))$ for $a.e.$ $s \in \mathbb{R}$. Relying on this convergence it can be easily checked that for $a.e.$ $(x,t) \in Q'_{0,T}$ there holds

$$
H_{\underline{g}_{\delta,j}}(x,t) \to H_{\underline{g}_\delta}(x,t) \quad \text{as } \ j \to \infty\,. \tag{7.44}
$$

Hence (7.42) follows by the Dominated Convergence theorem, combining (7.43)–(7.44). □

Proof of Theorem 3.6.. For every $\Omega' \subset\subset \Omega$ let λ_1 and λ_2 be the measures given in Proposition 7.2. Then the conclusion will follow by the arbitrariness of Ω' if we prove that

$$
\begin{aligned}
&\lambda_1(t) = D^s_+ u(t) \leq D^s_+ u_0, \\
&\lambda_2(t) = D^s_- u(t) \leq D^s_- u_0 \quad \text{in} \ \ \mathcal{M}(\Omega') \ \ (\text{for} \ \ a.e. \ t \in (0,T))\,,
\end{aligned}
\tag{7.45}
$$

$$
\begin{aligned}
&D^s_+ u(t_2) \leq D^s_+ u(t_1), \\
&D^s_- u(t_2) \leq D^s_- u(t_1) \quad \text{in} \ \ \mathcal{M}(\Omega') \ \ (\text{for} \ \ a.e. \ 0 < t_1 < t_2 < T)\,.
\end{aligned}
\tag{7.46}
$$

For every $\Omega' \subset\subset \Omega$, fix any $\zeta \in C^1([0,T]; C^1_c(\Omega'))$, $\zeta \geq 0$, $\zeta(T) = 0$ and $\zeta_t \in C_c(Q'_{0,T})$. Let us address the characterization of $\lambda_1(t)$ (see the first equality in (7.45)).

For every $\delta \in (0,1)$, let $\underline{g}_\delta$ be the function in (7.34) and let $\underline{G}_\delta$, $H_{\underline{g}_\delta}$ be the associated functions in (7.36) and (7.39), respectively. Then, for all $y \in]-\infty, 1)$,

$$
-1 \leq \underline{g}_\delta(y) \leq 0, \qquad \underline{g}_\delta(y) \to -1 \quad \text{as } \delta \to 0^+,
$$

whence, letting $\delta \to 0^+$ we easily obtain

$$
\underline{G}_\delta(u, u_x) \to -u_x \quad \text{in } \ L^1(Q'_{0,T})\,, \tag{7.47}
$$

$$G(u_0,u_{0x})\to -u_{0x}\quad \text{in } L^1(\Omega')\,, \tag{7.48}$$

$$\int_{Q'_{0,T}}\left\{-\underline{g}_\delta(\mathbf{a}(u,u_x))\,\zeta_x\,[\mathbf{a}(u,u_x)]_x+\underline{g}_\delta(\mathbf{a}(u,u_x))\,F_x\,\zeta\right\}\mathrm{d}x\mathrm{d}t$$
$$\to\int_{Q'_{0,T}}\{\zeta_x\,[\mathbf{a}(u,u_x)]_x-F_x\,\zeta\}\,\mathrm{d}x\mathrm{d}t \tag{7.49}$$

(here we have also used that $\mathbf{a}(u,u_x)<1$ *a.e.* in $Q'_{0,T}$). For every $\delta\in(0,1)$ and for *a.e.* $(x,t)\in Q'_{0,T}$ let $s_\delta(x,t)>0$ be the solution to the equation $\mathbf{a}(u(x,t),s)=1-\delta$. Then,

$$\mathbf{a}(u(x,t),s)>1-\delta\iff s>s_\delta(x,t) \tag{7.50}$$

(see assumption (H_3)), up to sets of zero Lebesgue measure there holds

$$\begin{aligned}&\left\{(x,t)\in Q'_{0,T}\,:\,\mathbf{a}(u(x,t),u_x(x,t))>1-\delta\right\}\\&\quad=\left\{(x,t)\in Q'_{0,T}\,:\,u_x(x,t)>s_\delta(x,t)\right\}\,,\end{aligned} \tag{7.51}$$

whereas by (1.10), for all $\delta>0$ small enough, we get

$$\left(\frac{C^+_{1,M}}{2\delta}\right)^{1/\sigma_1}\le\left(\frac{C^+_{1,M}}{\delta}-\max_{\{|z|\le M\}}\varphi_{1,M}(z)\right)^{1/\sigma_1}\le s_\delta(x,t)\le\left(\frac{C^-_{1,M}}{\delta}\right)^{1/\sigma_1}; \tag{7.52}$$

here $M>0$ is chosen so that

$$M\ge\|u\|_{L^\infty(Q'_{0,T})} \tag{7.53}$$

(see Remark (6.3)). Since $\underline{g}'_\delta(y)=\frac{1}{\delta}\chi_{\{1-\frac{1}{\delta}<y<1\}}$, by the very definition of the function $H_{\underline{g}_\delta}$ and (7.50) it follows that, for *a.e.* $(x,t)\in Q'_{0,T}$,

$$\begin{aligned}&H_{\underline{g}_\delta}(x,t)\\&=\chi_{\{\mathbf{a}(u,u_x)\ge1-\delta\}}\frac{u_t(x,t)}{\delta}\int_0^{u_x(x,t)}\chi_{\{\mathbf{a}(u(x,t),s)>1-\delta\}}\partial_z\mathbf{a}(u(x,t),s)\,\mathrm{d}s\\&=\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{u_t(x,t)}{\delta}\int_{s_\delta(x,t)}^{u_x(x,t)}\partial_z\mathbf{a}(u(x,t),s)\,\mathrm{d}s\\&=\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{[\mathbf{a}(u(x,t),u_x(x,t))]_x}{\delta}\int_{s_\delta(x,t)}^{u_x(x,t)}\partial_z\mathbf{a}(u(x,t),s)\,\mathrm{d}s\\&\quad+\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{F(x,t)}{\delta}\int_{s_\delta(x,t)}^{u_x(x,t)}\partial_z\mathbf{a}(u(x,t),s)\,\mathrm{d}s=:H^{(1)}_{\underline{g}_\delta}(x,t)+H^{(2)}_{\underline{g}_\delta}(x,t)\,.\end{aligned} \tag{7.54}$$

In view of assumption (H_3) (with M as in (7.53) and $\sigma_0 > 1$) and (7.52), for *a.e.* $(x,t) \in Q'_{0,T}$ we have

$$\begin{aligned}\left|H^{(1)}_{\underline{g}_\delta}(x,t)\right| &\leq \frac{D_M}{\delta}\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\,|[\mathbf{a}(u(x,t),u_x(x,t)]_x|\int_{\left(\frac{C^+_{1,M}}{2\delta}\right)^{1/\sigma_1}}^{+\infty}\frac{1}{(1+s)^{\sigma_0}}\mathrm{d}s\\ &\leq \frac{\tilde{D}_M}{\delta}\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\,|[\mathbf{a}(u(x,t),u_x(x,t)]_x|\;\delta^{(\sigma_0-1)/\sigma_1}\\ &\leq \chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{\left([\mathbf{a}(u(x,t),u_x(x,t))]_x\right)^2}{2\delta}\\ &\quad+\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{\tilde{D}^2_M}{2}\delta^{\frac{2\sigma_0-2-\sigma_1}{\sigma_1}}\end{aligned} \tag{7.55}$$

(here we have used Young's inequality). Concerning the second term in the right-hand side of (7.54), there holds

$$H^{(2)}_{\underline{g}_\delta}(x,t) = 0 \quad \text{if } F \equiv 0 \text{ in } Q_{0,T}\,; \tag{7.56}$$

otherwise, if $F \not\equiv 0$, for *a.e.* $(x,t) \in Q'_{0,T}$ we have

$$\begin{aligned}\left|H^{(2)}_{\underline{g}_\delta}(x,t)\right| &\leq \chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\frac{\|F\|_\infty}{\delta}\int_{\left(\frac{C^+_{1,M}}{2\delta}\right)^{1/\sigma_1}}^{+\infty}\frac{1}{(1+s)^{\sigma_0}}\mathrm{d}s\\ &\leq \tilde{D}_M\chi_{\{u_x(x,t)>s_\delta(x,t)\}}\|F\|_\infty\delta^{\frac{\sigma_0-1}{\sigma_1}-1}\\ &\leq \tilde{D}_M\chi_{\left\{u_x(x,t)>\left(C^+_{1,M}/2\delta\right)^{1/\sigma_1}\right\}}\|F\|_\infty\delta^{\frac{\sigma_0-\sigma_1}{\sigma_1}}\left(\frac{1}{\delta}\right)^{1/\sigma_1}\\ &\leq \bar{D}^{(1)}_{M,F}\delta^{\frac{\sigma_0-\sigma_1}{\sigma_1}}u_x(x,t)\,\chi_{\left\{u_x(x,t)>\left(C^+_{1,M}/2\delta\right)^{1/\sigma_1}\right\}}\end{aligned}$$

(see also (7.51) and (7.52)). By the above inequality, since $\sigma_0 - \sigma_1 \geq 0$ (see (H3)) and $u_x \in L^1(Q'_{0,T})$, in the case $F \not\equiv 0$ from the Dominated Convergence theorem it follows that

$$\begin{aligned}\left|H^{(2)}_{\underline{g}_\delta}\right| &\leq \bar{D}^{(1)}_{M,F}\delta^{\frac{\sigma_0-\sigma_1}{\sigma_1}}u_x\chi_{\left\{(x,t)\in Q'_{0,T}:u_x(x,t)>\left(C^+_{1,M}/2\delta\right)^{1/\sigma_1}\right\}}\\ &\to 0 \text{ in } L^1(Q'_{0,T}) \text{ as } \delta\to 0^+.\end{aligned} \tag{7.57}$$

Combining inequality (7.37) with (7.54)–(7.55) gives

$$\begin{aligned}&\frac{1}{2\delta}\int_{Q'_{0,T}}\left([\mathbf{a}(u,u_x)]_x\right)^2\chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\,\zeta\,\mathrm{d}x\mathrm{d}t\\ &\leq \int_{Q'_{0,T}}\underline{G}_\delta(u,u_x)\,\zeta_t\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_2(t)\right) \mathrm{d}t + \frac{\tilde{D}_M^2}{2}\delta^{\frac{2\sigma_0-2-\sigma_1}{\sigma_1}} \int_{Q'_{0,T}} \chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\, \zeta \,\mathrm{d}x\mathrm{d}t \\
&+\int_{Q'_{0,T}} H^{(2)}_{\underline{g}_\delta}\, \zeta \,\mathrm{d}x\mathrm{d}t + \int_{\Omega'} \underline{G}_\delta(u_0, u_{0x})\, \zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\, dD^s_- u_0 \\
&-\int_{Q'_{0,T}} \underline{g}_\delta(\mathbf{a}(u,u_x))\, \zeta_x\, [\mathbf{a}(u,u_x)]_x \,\mathrm{d}x\mathrm{d}t + \int_{Q'_{0,T}} \underline{g}_\delta(\mathbf{a}(u,u_x))\, F_x\, \zeta \,\mathrm{d}x\mathrm{d}t\,, \quad (7.58)
\end{aligned}
$$

where (see (7.51) and (7.52))

$$
\begin{aligned}
&\delta^{\frac{2\sigma_0-2-\sigma_1}{\sigma_1}} \int_{Q'_{0,T}} \chi_{\{\mathbf{a}(u,u_x)>1-\delta\}}\, \zeta \,\mathrm{d}x\mathrm{d}t \le \|\zeta\|_{L^\infty(Q'_{0,T})} \delta^{\frac{2\sigma_0-2-\sigma_1}{\sigma_1}} \left|\{\mathbf{a}(u,u_x) > 1-\delta\}\right| \\
&\le \|\zeta\|_{L^\infty(Q'_{0,T})} \delta^{\frac{2\sigma_0-2-\sigma_1}{\sigma_1}} \left|\left\{(x,t)\in Q'_{0,T} \,:\, u_x > \left(C^+_{1,M}/2\delta\right)^{1/\sigma_1}\right\}\right| \\
&\le C_{\zeta,M}\delta^{\frac{2\sigma_0-1-\sigma_1}{\sigma_1}} \to 0 \quad \text{as } \delta \to 0^+\,, \quad (7.59)
\end{aligned}
$$

as $2\sigma_0 - 1 - \sigma_1 > 0$ (see (H3)). By (7.47)–(7.49) and (7.56), (7.57), (7.59), letting $\delta \to 0^+$ in (7.58) gives

$$
\begin{aligned}
&\int_{Q'_{0,T}} \{-u_x\zeta_t + [\mathbf{a}(u,u_x)]_x\, \zeta_x - F_x\, \zeta\} \,\mathrm{d}x\mathrm{d}t + \int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_2(t)\right) \mathrm{d}t \\
&\quad -\int_{\Omega'} u_{0x}\, \zeta(x,0)\,\mathrm{d}x + \int_{\Omega'} \zeta(x,0)\, dD^s_- u_0 \ge 0\,. \quad (7.60)
\end{aligned}
$$

Since $\lambda_1(t) - \lambda_2(t) = D^s u(t)$ in $\mathcal{M}(\Omega')$ for *a.e.* $t \in (0,T)$ (see (7.10)), summing up (7.60) and (3.1) we obtain

$$
\int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_1(t)\right) \mathrm{d}t \ge -\int_{\Omega'} \zeta(x,0)\, dD^s_+ u_0\,. \quad (7.61)
$$

Analogously, relying on (7.38) and arguing in a similar way, it can be checked that

$$
\int_0^T \left(\int_{\Omega'} \zeta_t(t) d\lambda_2(t)\right) \mathrm{d}t \ge -\int_{\Omega'} \zeta(x,0)\, dD^s_- u_0 \quad (7.62)
$$

for every $\zeta \in C^1([0,T]; C^1_c(\Omega'))$, $\zeta \ge 0$, $\zeta(T) = 0$ and $\zeta_t \in C_c(Q'_{0,T})$.

Fix any $\tau \in (0,T)$ and set

$$
\eta_j(t) = \chi_{\{0\le t\le\tau\}} + j(\tau + 1/j - t)\chi_{\{\tau<t<\tau+1/j\}} \qquad (j \in \mathbb{N})\,.
$$

Notice that for every nonnegative $\rho \in C^1_c(\Omega')$ and j large enough, by standard approximation arguments the test function $\zeta_j(x,t) = \eta_j(t)\rho(x)$ is an admissible choice in (7.61)–(7.62). Then we have

$$
j\int_\tau^{\tau+1/j} \left(\int_{\Omega'} \rho\, d\lambda_1(t)\right) \mathrm{d}t \le \int_{\Omega'} \rho\, dD^s_+ u_0\,,
$$

$$j \int_{\tau}^{\tau+1/j} \left(\int_{\Omega'} \rho \, d\lambda_2(t) \right) \mathrm{d}t \le \int_{\Omega'} \rho \, dD_-^s u_0 \, .$$

Hence, letting $j \to \infty$ in the above inequalities gives for $a.e.\ \tau \in (0, T)$ and for every ρ as above

$$\int_{\Omega'} \rho \, d\lambda_1(\tau) \le \int_{\Omega'} \rho \, dD_+^s u_0, \qquad \int_{\Omega'} \rho \, d\lambda_2(\tau) \le \int_{\Omega'} \rho \, dD_-^s u_0 \, .$$

Thus, the nonnegative measures $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are absolutely continuous with respect to $D_+^s u_0$ and $D_-^s u_0$, respectively, therefore singular with respect to the Lebesgue measure over Ω and, in addition, mutually singular. Since $D^s u(\tau) = \lambda_1(\tau) - \lambda_2(\tau)$ for $a.e.\ \tau \in (0, T)$ (see (7.10)), by the uniqueness of the Jordan decomposition of $D^s u(t)$, we get both equalities in (7.45).

Finally, in order to prove, for $a.e.\ 0 < \tau_1 < \tau_2 < T$, choosing (by standard approximation arguments) in (7.61)–(7.62) the test functions $\eta_j(x, t) = \tilde{\eta}_j(t)\rho(x)$, with $\rho \in C_c^1(\Omega)$, $\rho \ge 0$, and

$$\tilde{\eta}_j(t) = j(t - \tau_1 + 1/j)\chi_{\{\tau_1 - 1/j \le t \le \tau_1\}} + \chi_{\{\tau_1 < t < \tau_2\}} + j(\tau_2 + 1/j - t)\chi_{\{\tau_2 \le t \le \tau_2 + 1/j\}}$$

($j \in \mathbb{N}$, large enough), we get

$$j \int_{\tau_2}^{\tau_2+1/j} \left(\int_{\Omega'} \rho \, dD_+^s u(t) \right) \mathrm{d}t \le j \int_{\tau_1 - 1/j}^{\tau_1} \left(\int_{\Omega'} \rho \, dD_+^s u(t) \right) \mathrm{d}t$$
$$j \int_{\tau_2}^{\tau_2+1/j} \left(\int_{\Omega'} \rho \, dD_-^s u(t) \right) \mathrm{d}t \le j \int_{\tau_1 - 1/j}^{\tau_1} \left(\int_{\Omega'} \rho \, dD_-^s u(t) \right) \mathrm{d}t$$

(here we have used the identifications $\lambda_1(t) = D_+^s u(t)$ and $\lambda_2(t) = D_-^s u(t)$). Therefore the conclusion follows from the above inequalities, in the limit as $j \to \infty$. □

Proof of Theorem 3.7. We shall only address claim (i), the proof of (ii) being analogous. To this aim, let us preliminarily observe that the condition $\mathbf{a}(u, u_x) \in L^2(\tau, T; H^1(\Omega))$ for all $\tau \in (0, T)$ (see Definition 3.1–(ii)) ensures that

$$\mathbf{a}(u(t), u_x(t)) \in H^1(\Omega) \quad \text{for } a.e.\ t \in (0, T) \, . \tag{7.63}$$

Next, fix any open interval $\Omega' \subset\subset \Omega$ and observe that

$$u(t) \in BV(\Omega') \quad \text{for } a.e.\ t \in (0, T) \, , \tag{7.64}$$

as $u \in L_w^1(0, T; BV_{\mathrm{loc}}(\Omega))$ (see Definition 3.1). Let us fix any $t \in (0, T)$ as in (7.63)–(7.64). By (7.64) there exists $M \equiv M(\Omega', t) > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega')} \le M \, ,$$

whereas from (7.63) there exists $C \equiv C(t) > 0$ such that

$$|\, [\mathbf{a}(u(t), u_x(t))](x_2) - [\mathbf{a}(u(t), u_x(t))](x_1) \,| \le C\sqrt{|x_2 - x_1|} \quad \text{for all } x_1, x_2 \in \Omega \, ; \tag{7.65}$$

in the above inequality we have identified the function $\mathbf{a}(u(t), u_x(t))$ with its continuous representative. Arguing by contradiction, assume that

$$D^s_+ u(t) \neq 0 \quad \text{in } \mathcal{M}(\Omega') .$$

Then, in view of Proposition 3.1, there exists $x_0 \in \Omega'$ such that $[\mathbf{a}(u(t), u_x(t))](x_0) = 1$, and choosing in (7.65) $x_2 = x_0$ gives

$$1 - [\mathbf{a}(u(t), u_x(t))](x) = |\,1 - [\mathbf{a}(u(t), u_x(t))](x)\,| \leq C\sqrt{|x_0 - x|} \quad \text{for all } x \in \Omega' . \tag{7.66}$$

Moreover, by the continuity of the function $x \mapsto \mathbf{a}(u(t), u_x(t))$, there exists $\delta > 0$ such that $[\mathbf{a}(u(t), u_x(t))](x) > 0$ for all $x \in I_\delta(x_0) := \Omega' \cap (x_0 - \delta, x_0 + \delta)$, whence $u_x(\cdot, t) > 0$ *a.e.* in $I_\delta(x_0)$ (recall that the map $\xi \mapsto \mathbf{a}(z, \xi)$ is nondecreasing and $\mathbf{a}(z, 0) = 0$ for all $z \in \mathbb{R}$). Combining (7.66) with assumption (H_4)–(i), we get

$$\frac{C^+_{3,M}}{(1 + |u_x(x,t)|)^\sigma} \leq 1 - [\mathbf{a}(u(x,t), u_x(x,t))] \leq C\sqrt{|x_0 - x|} \quad \text{for } a.e.\ x \in I_\delta(x_0) ,$$

whence

$$1 + |u_x(x,t)| \geq \frac{\bar{C}}{|x - x_0|^{1/(2\sigma)}} \quad \text{for } a.e.\ x \in I_\delta(x_0)$$

where $\bar{C} := \left(\frac{C^+_{3,M}}{C}\right)^{\frac{1}{\sigma}}$. Since $1/(2\sigma) \geq 1$ and $u_x(t) \in L^1(\Omega')$, we have reached a contradiction and the conclusion follows. □

Proof of Theorem 3.8. In view of assumption (H_3), for any initial data $u_0 \in BV(\Omega)$ by Theorem 3.6 strong solutions u to $(P^{u_0}_{\alpha,\beta})$ satisfy the monotonicity properties

$$D^s_\pm u(\tau) \leq D^s_\pm u(t) \leq D^s_\pm u_0 \quad \text{in } \mathcal{M}(\Omega) \tag{7.67}$$

for $a.e.\ 0 < t < \tau < T$.

Fix any open interval $\Omega' \subset\subset \Omega$. Since $u \in L^\infty_w(0, T; BV_{\text{loc}}(\Omega))$ (see Theorem 3.4), there exists $M_1 \equiv M_1(\Omega') > 0$ such that

$$\|u\|_{L^\infty(\Omega' \times (0,T))} \leq M_1 . \tag{7.68}$$

Next, for any $j \in \mathbb{N}$ large enough and $\tau \in (0, T)$, set

$$h_j(t) = \chi_{[0,\tau - j^{-1}]}(t) + j(\tau - t)\chi_{(\tau - j^{-1}, \tau]} \qquad (t \in [0, T]) .$$

For any $\rho \in C^1_c(\Omega')$, choosing $\zeta(x,t) = \rho(x)h_j(t)$ in equality (3.1) and letting $j \to \infty$, by a routine proof we get

$$\int_{\Omega'} \rho(x)\, dDu(\tau) - \int_{\Omega'} \rho(x)\, dDu_0 = - \int_{Q_{0,\tau}} [\mathbf{a}(u, u_x)]_x \rho'(x)\, \mathrm{d}x\mathrm{d}t + \int_{Q_{0,\tau}} F_x(x,t)\, \rho(x)\, \mathrm{d}x\mathrm{d}t . \tag{7.69}$$

Let $\mathcal{N} \subset (0, T)$ be any null set up to which (7.67) and (7.69) are satisfied.

Arguing by contradiction, assume that there exist $\tau \in (0, T) \setminus \mathcal{N}$ and $x_0 \in \Omega'$ such that $x_0 \in \operatorname{supp} D^s_+ u(\tau)$ (the case where $x_0 \in \operatorname{supp} D^s_-(x_0)$ is completely analogous). By (7.67) it follows that $x_0 \in \operatorname{supp} D^s_+ u(t)$ for all $t \in (0, \tau) \setminus \mathcal{N}$, whence (see Proposition 3.1)

$$[\mathbf{a}(u(t), u_x(t))](x_0) = 1 \quad \text{for all } t \in (0, \tau) \setminus \mathcal{N} \tag{7.70}$$

(here we have identified the function $\mathbf{a}(u(t), u_x(t))$ with its continuous representative). Let $V : \Omega' \to \mathbb{R}$ be the function defined by setting

$$V(x) := \int_0^\tau [\mathbf{a}(u(x,t), u_x(x,t))] \, dt \qquad (x \in \Omega') .$$

By equality (7.69) it follows that

$$V_{xx} = Du(\tau) - Du(0) - \int_0^\tau F_x(\cdot, t) \, dt \quad \text{in } \mathcal{D}'(\Omega') .$$

Since the right-hand side in the above equality is a finite Radon measure over Ω', it can be easily checked that $V_x \in BV(\Omega')$. In particular, this implies that there exists $M_2 = M_2(\Omega', \tau) > 0$ such that $\|V_x\|_{L^\infty(\Omega')} \le M_2$. Therefore, for every $x \in \Omega'$ there holds

$$\tau - V(x) = |\tau - V(x)| = |V(x_0) - V(x)| \le M_2 |x - x_0| \tag{7.71}$$

(here we have used that $V(x_0) = \tau \ge V(x)$ for all $x \in \Omega'$; see (7.70) and recall that $\mathbf{a}(z, \xi) \le 1$). On the other hand, since $|u| \le M_1$ *a.e.* in $\Omega' \times (0, T)$, by assumption (H_3) we have

$$\begin{aligned} 1 - \mathbf{a}(u, u_x) \ge 1 - \mathbf{a}(u, [u_x]^+) &\ge \frac{C^+_{1,M_1}}{C_{M_1} + ([u_x]^+)^{\sigma_1}} \\ &\ge \frac{C^+_{1,M_1}}{2(C_{M_1}^{1/\sigma_1} + [u_x]^+)^{\sigma_1}} \quad a.e. \text{ in } \Omega' \times (0, T) , \end{aligned}$$

for some $C_{M_1} > \|\varphi_{1,M}\|_{L^\infty((-M_1, M_1))}$. In the first inequality of the above estimate we have also used that for every $z \in \mathbb{R}$ there holds $\mathbf{a}(z, 0) = 0$, and $\mathbf{a}(z, \xi) \le 0$ for all $\xi \le 0$. In the last one, we have used the concavity of the function $s \mapsto s^{\sigma_1}$.

Thus, for $a.e.$ $x \in \Omega'$ there holds

$$\begin{aligned} \tau - V(x) &= \int_0^\tau [1 - \mathbf{a}(u(x,t), u_x(x,t))] \, dt \\ &\ge \frac{C^+_{1,M_1}}{2} \int_0^\tau \frac{1}{(C_{M_1}^{1/\sigma_1} + [u_x(x,t)]^+)^{\sigma_1}} \, dt \\ &\ge \frac{\tau \, C^+_{1,M_1}}{2} \cdot \left(C_{M_1}^{1/\sigma_1} + \frac{1}{\tau} \int_0^\tau [u_x(x,t)]^+ \, dt \right)^{-\sigma_1} \end{aligned}$$

(here we have used the Jensen inequality, as the map $\xi \mapsto (C_{M_1}^{\frac{1}{\sigma_1}} + \xi)^{-\sigma_1}$ is convex on $[0, \infty)$). Combining the above inequality with (7.71), it follows that there exists $C = C(M_1, M_2) > 0$ such that, for $a.e.\ x \in \Omega'$, there holds

$$\frac{1}{\tau}\int_0^\tau [u_x(x,t)]^+ \, dt \geq \frac{C\tau^{\frac{1}{\sigma_1}}}{|x - x_0|^{\frac{1}{\sigma_1}}} - C_{M_1}^{1/\sigma_1}.$$

Since $\sigma_1 \in (0, 1]$, this proves that $u_x^+ \notin L^1(\Omega' \times (0,\tau))$, a contradiction as $u \in L_w^\infty(0, T; BV(\Omega'))$. □

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