Preprints of the 20th World Congress
The International Federation of Automatic Control
Toulouse, France, July 9-14, 2017

State Feedback Stabilization of Linear Systems with Unknown Input Time Delay

Filippo Cacace * **Francesco Conte** ** **Alfredo Germani** ***

* *Università Campus Bio-Medico di Roma, Roma, Italy, (e-mail: f.cacace@unicampus.it).*
** *DITEN, Università degli studi di Genova, Genova, Italy, (e-mail: fr.conte@unige.it).*
*** *DISIM, Università degli studi dell'Aquila, L'Aquila, Italy, (e-mail: alfredo.germani@univaq.it).*

Abstract: This paper investigates the control problem of linear systems affected by an unknown constant input delay by means of a finite-dimensional state feedback. The proposed solution extends an approach used in the case of known delays by means of a suitably developed delay identifier. A more general result about the convergence to zero of the controlled system when the delay estimation error only converges to some neighborhood of zero is provided. Numerical examples show the effectiveness of the proposed approach.

Keywords: State feedback, time delay systems, time delay estimation, feedback stabilization, time delay, linear systems.

1. INTRODUCTION

The control problem of linear and nonlinear systems affected by input delay has been widely studied in recent years (Bresch-Pietri and Krstic, 2013; Krstic, 2009, 2010; Cacace et al., 2016). Most of the proposed approaches extend the classic infinite-dimensional predictor approach (Manitius and Olbrot, 1979; W. Michiels and Ztek, 2010) or, more recently, use finite-dimensional control laws (Cacace et al., 2014, 2015; Zhou et al., 2012; Yoon and Lin, 2013; Zhou, 2014). All of them assume the precise knowledge of the input delay. When the delay is unknown, one may resort to robust controllers (Fridman, 2014) but losing useful properties such as the exponential convergence. An alternative approach is to couple the controller with a delay identifier. However, the delay identification problem is challenging and very few contributions are devoted to this problem. The methods proposed in Belkoura (2005) and Belkoura et al. (2009) concern with parametric identification of delay systems with possibly unknown delays for first order linear systems. In Hetel et al. (2011) a Linear Matrix Inequality (LMI) based controller that uses an approximate knowledge of a time-varying delay is realized. The paper Polyakov et al. (2013) proposes an interval observer for an input time-varying delay to realize an adaptive output stabilization of linear systems using an infinite-dimensional predictor controller. In Bresch-Pietri and Krstic (2010) a delay-adaptive controller for linear systems with a single constant delay is designed by coupling an infinite-dimensional predictor with a delay identifier. This approach has been extended to uncertain systems in Bresch-Pietri et al. (2012) and nonlinear systems in Bresch-Pietri and Krstic (2014).

In this paper, we consider a finite-dimensional predictor controller developed for linear systems with known input delays (Cacace et al., 2014). The idea is to use the same control law but replacing the real delay with an estimate. As first contribution, we study the robustness of the considered controller with respect to the delay estimation error dynamics. Supposing the conditions to exponentially stabilize the system, given the knowledge of the true delay, to be satisfied, we prove that: 1) if the estimate converges to the true delay value, the controlled system is exponentially stable, 2) if the delay estimate definitely belongs to a finite interval containing the true value, the exponentially stability of the controlled system is still guaranteed. As second contribution, we propose a delay identifier that, under suitable hypotheses, converges to this interval and guarantees the exponential convergence to zero of the controlled system.

Notation. Given $x \in \mathbb{R}^n$, $\|x\|$ is the euclidean norm. $\mathcal{C}(A;B)$ denotes the set of continuous functions that map A into B with the uniform convergence norm, denoted by $\|\cdot\|_\infty$. Given an integer n and a positive real number δ, $\mathcal{C}^n_{\bar\delta} = \mathcal{C}([-\delta,0];\mathbb{R}^n)$. $I_{\bar\delta}$ is the interval $[0,\bar\delta]$. The subscript μ in $f_\mu(t)$, where $f \in \mathcal{C}([a,b];\mathbb{R}^m)$ for some integer m and real numbers $b > a \geq 0$ denotes that f depends on the function $\mu \in \mathcal{C}([a,b]; I_{\bar\delta})$. $W^{1,2}$ indicates the space of absolutely continuous functions form $[-\delta,0]$ into $\mathbb{R}^n$. $\sigma(A)$ is the spectrum of a square real matrix A. I_n is the identity matrix in $\mathbb{R}^n$. Given a linear operator T which maps a normed space H into itself, $\|T\|_H$ denotes the induced operator norm $\sup_{x\in H} \|Tx\|_H/\|x\|_H$. In the special case of a matrix A, $\|A\|$ is the euclidean induced matrix norm. Finally, a function $\mathcal{T} : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it is zero at zero, continuous and strictly increasing.

2. PROBLEM FORMULATION

Consider a linear system with the following form:

$$\dot{x}(t) = Ax(t) + Bu(t-\delta) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t-\delta) \in \mathbb{R}^p$ is the delayed control input. The delay δ is unknown and belongs to the interval $I_{\bar{\delta}}$. The state $x(t)$ is supposed to be available at time t. Under the hypothesis that the pair (A, B) is controllable, we consider the problem of constructing a stabilizing control law with a delay dependent gain matrix, similarly to Cacace et al. (2014). In the present paper, since the delay is unknown, we suppose to have an estimate $\hat{\delta}(t)$. Therefore, we consider a feedback control law with the following form:

$$u(t) = -K(\hat{\delta}(t))x(t) \tag{2}$$

where $K(\hat{\delta}(t)) \in \mathbb{R}^{p\times n}$. We assume that the control law (2) starts operating at time $t = -\delta$. Therefore, the closed-loop system has the form of a state time-delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BK(\hat{\delta}(t-\delta))x(t-\delta), \quad t \geq 0 \\ x(t) &= \phi(t), \qquad t \in [-\delta, 0]. \end{aligned} \tag{3}$$

where $\phi \in \mathcal{C}^n_\delta$ is the so called *preshape function*.

The objective of this paper is to design a feedback control law with the form in (2) coupled with a time-delay identifier, which computes the estimate $\hat{\delta}(t)$, in order to ensure the exponential stability of the system state.

The following definition, introduced in Cacace et al. (2014) and Fridman (2014), will be useful in the sequel.

Definition 1. (Exponential stability). For a given real number $\alpha > 0$, system (3) is said to be α-*exp* stable if there exist $\gamma > 0$ such that

$$||x(t)|| \leq e^{-\alpha t}\gamma||\phi||_\infty, \quad \forall t \geq 0, \ \ \forall \phi \in \mathcal{C}^n_\delta. \tag{4}$$

System (3) is exponentially stable if there exists a real number $\alpha > 0$ such that (4) holds true.

3. THE FEEDBACK LAW

We propose the following delay dependent control law:

$$u(t) = -K(\hat{\delta}(t))x(t), \quad \text{with } \ K(\hat{\delta}(t)) = \bar{K}e^{\bar{A}\hat{\delta}(t)} \tag{5}$$

where $\bar{A} = A - B\bar{K}$, with $\bar{K} \in \mathbb{R}^{p\times n}$ is such that $\bar{A}$ is Hurwitz with arbitrarily assigned eigenvalues (possible in virtue of the controllability of the pair (A, B)). The feedback control law (5) has the same form proposed in Cacace et al. (2014), where the time-delay δ is known. We report in the next Lemma a key result of Cacace et al. (2014) (Theorem 1) useful to the present paper.

Lemma 1. Consider system (1) with the control law (5). Assume that $\hat{\delta}(t) = \delta \leq \bar{\delta}$ for all $t \geq -\delta$. If there exists a real number $\alpha > 0$ such that

$$\int_0^{\bar{\delta}} \|\bar{K}e^{\bar{A}t}B\|e^{\alpha t}dt \leq 1, \tag{6}$$

then, the closed loop system (3) is $\tilde{\alpha}$-*exp* stable, with $\tilde{\alpha} \geq \alpha$.

Lemma 1 states that, when condition (6) holds true, if the delay identification error $\epsilon(t) = \delta - \hat{\delta}(t)$ is null, i.e. $\hat{\delta}(t) = \delta$, the controlled system is exponentially stable with rate $\tilde{\alpha} > 0$. In the present paper, we are interested in studying what happens when the delay estimate $\hat{\delta}(t)$ is computed by a properly designed delay identifier. The following theorems give the conditions on the delay identification error to keep the exponential stability of the controlled system.

Theorem 2. Consider system (1) with the control law (5). Assume that condition (6) is satisfied for a positive real α. If $|\epsilon(t)|$ is uniformly bounded for all $t \geq -\delta$ and $\lim_{t\to+\infty} \epsilon(t) = 0$, then system (3) is exponentially stable.

Theorem 3. Consider system (1) with the control law (5). Assume that condition (6) is satisfied for a positive real α and that $|\epsilon(t)|$ is uniformly bounded for all $t \geq -\delta$. Then, there exists $\bar{\epsilon} > 0$ such that, if for some $t^* \geq 0$ $|\epsilon(t)| \leq \bar{\epsilon}$, $\forall t \geq t^*$, then system (3) is exponentially stable.

The proofs to these theorems are given in Section 5. Theorem 2 is the first main result of the paper. It states that, once designed a feedback control law able to make exponentially stable the system when the delay identification error is zero, it is possible to keep the exponential stability if the delay identification converges to zero.

Theorem 3 says that it is possible to keep the exponential stability of the system even if the delay identifier is only able to definitively drive the identification error into a sufficiently small interval around zero.

4. THE DELAY IDENTIFIER

As discussed in last section, in order to obtain the exponential stability of system (1) by using the control law (5), we need a delay identifier able to drive the estimate $\hat{\delta}(t)$ toward the real value δ. To this aim, it is required that the delay is *identifiable from the state measurements*. It is important to stress that the system dynamics depends on the time-delay since the control input is delayed. As a consequence, the delay identifiability also depends on the control law and we therefore provide a definition with reference to the closed-loop system (3).

Definition 2. The delay δ in the closed-loop system (3) is identifiable from the state if, for any non-zero initial condition, it is not possible to have two delays δ_1 and δ_2, $\delta_1 \neq \delta_2$, such that $x_1(t) = x_2(t)$ for all $t > 0$, where $x_1(t)$ and $x_2(t)$ are the state vector trajectories associated to δ_1 and δ_2, respectively.

The delay identifiability assumption can be tested off-line by analytic means or by simulations using copies of the system with distinct delays.

The delay identifier that we propose is a piecewise constant function, i.e. $\hat{\delta}(t)$ is constant in time intervals of uniform length $\Delta > \bar{\delta}$. Let $t_k = -\delta + k\Delta$, $k = 0, 1, 2, \ldots$ so that $[-\delta, \infty)$ is partitioned into the intervals $[t_k, t_{k+1})$. Given an initial estimate $\hat{\delta}_0 \in I_{\bar{\delta}}$ at $t = -\delta = t_0$, the piece-wise constant delay estimate $\hat{\delta}(t)$ is defined as

$$\hat{\delta}(t) = \begin{cases} d_{k-1}(t_k), & t \in [t_k, t_{k+1}), \ k > 0, \\ \hat{\delta}_0, & t \in [t_0, t_1), \end{cases} \tag{7}$$

where $d_k(t)$ are *temporary* delay estimates, each one defined in $\ell_k = [t_k, t_{k+1}]$ with the following dynamics:

$$\begin{aligned} \dot{d}_k(t) &= \frac{1}{\eta + \beta\|\mathcal{S}_k(t)\|}\mathrm{Proj}_{[0,\delta]}\left(r_k^T(t)\mathcal{S}_k(t)\right) \\ d_k(t_0) &= \hat{\delta}_0 \text{ for } k = 0, \\ d_k(t_k) &= d_{k-1}(t_k) \text{ for } k \geq 1, \end{aligned} \tag{8}$$

where:

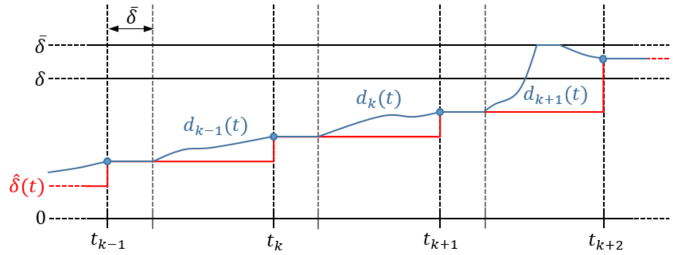

Fig. 1. Example of the delay identification dynamics.

- the standard projection operator is given by

$$\mathrm{Proj}_{[0,\bar{\delta}]}(\tau) = \begin{cases} 0, & d_k(t) = 0 \text{ and } \tau < 0 \\ 0, & d_k(t) = \bar{\delta} \text{ and } \tau > 0 \\ \tau, & \text{otherwise} \end{cases} . \quad (9)$$

- $\eta, \beta > 0$ are tuning parameters.
- $r_k(t)$ is the difference

$$r_k(t) = x(t) - \xi_k(t) \quad (10)$$

between the system state and an observer $\xi_k(t)$, defined as $\xi_k(t) = x(t)$ in $[t_k - \bar{\delta}, t_k + \bar{\delta}]$, and

$$\dot{\xi}_k(t) = A\xi_k(t) - BK(d_k(t_k))\xi_k(t - d_k(t)), \quad (11)$$

in $[t_k + \bar{\delta}, t_{k+1}]$.
- $\mathcal{S}_k(t)$ is the sensitivity of $\xi_k(t)$ with respect to the delay estimate $d_k(t)$, defined as

$$\mathcal{S}_k(t) = \left.\frac{\partial \xi_k(t)}{\partial d_k}\right|_{d_k = d_k(t)}, \quad (12)$$

and computed by integrating in $[t_k + \bar{\delta}, t_{k+1}]$

$$\dot{\mathcal{S}}_k(t) = A\mathcal{S}_k(t) - BK(d_k(t_k)) \cdot \Big(\mathcal{S}_k(t - d_k(t)) - \dot{\xi}_k(t - d_k(t))\Big), \quad (13)$$

with initial condition $\mathcal{S}_k(t) = 0$ in $[t_k - \bar{\delta}, t_k + \bar{\delta}]$. In (13), $\dot{\xi}_k(t - d_k(t))$ is computed at time t by using (11).

Figure 1 illustrates the dynamics of the delay identifier. Within each interval ℓ_k, the temporary estimate d_k evolves following (8). In particular, within the initial time interval $[t_k, t_k + \bar{\delta}]$ it is constant and equal to the last value reached by the preceding temporary estimate d_{k-1} (notice that $\xi_k(t) = x(t)$ and therefore $r_k(t) = 0$ in this interval, see (11)). Then, d_k is driven by a differential equation depending on the residual r_k and the sensitivity $\mathcal{S}_k$. The temporary delay estimate reached at the end of the interval is used to update the delay estimate $\hat{\delta}(t)$ and to initialize the following temporary estimate d_{k+1}.

As said in the notation description at the end of Section 1, subscript μ in $f_\mu(t)$ denotes that a function f depends on the function μ. In (8)–(13), $r_k(t)$, $\xi_k(t)$ and $\mathcal{S}_k(t)$ depends on the temporary estimates $d_k(t)$. The notation rule should be therefore applied to these quantities. However, in order to improve the paper readability, in these cases, the subscript d_k has been substituted with a simple k.

We provide now a motivation to the piece-wise structure of the proposed delay identifier. The idea is to reach the real delay using an anti-gradient based approach. More specifically, to direct the delay estimate along the opposite direction of the gradient of the difference between the real state x and the state $\xi_{\hat{\delta}}$ of a copy of the system, generated using the estimated delay $\hat{\delta}$, i.e. using an update equation of the form $\dot{\hat{\delta}} = -\gamma \nabla_{\hat{\delta}}(x - \xi_{\hat{\delta}})$. As done for example in Cacace et al. (2016), realizing this idea requires the computation of a sensitivity vector $\mathcal{S}_{\hat{\delta}} = \nabla_{\hat{\delta}} \xi_{\hat{\delta}}$. Unfortunately, because of the use of the control law (5), which depends on the estimated delay $\hat{\delta}$, the computation of $\mathcal{S}_{\hat{\delta}}$ needs the knowledge of the real delay δ, which is obviously not available. This does not hold if the estimation delay used in (5) is constant. Therefore, the problem has been solved as proposed in (8)–(13), by partitioning the time into the intervals ℓ_k, keeping constant the delay estimate $\hat{\delta}$ used in the control law within each interval, applying the anti-gradient based method to a temporary delay estimate d_k, and updating $\hat{\delta}$ at the end of each interval.

Next theorem provides a sufficient condition for the convergence to zero of the delay identification error $\epsilon(t)$.

Theorem 4. Assume that, for all $k = 0, 1, 2, \ldots$, there exist a function $\mathcal{T}_k : [0, \Delta - \bar{\delta}] \to \mathbb{R}_+$ of class $\mathcal{K}$ and two positive constants η, β such that

$$\int_{t_k + \bar{\delta}}^{t} \frac{\mathcal{S}^T_{\tilde{d}_k}(\tau)\mathcal{S}_k(\tau)}{\eta + \beta \|\mathcal{S}_k(\tau)\|} d\tau = \mathcal{T}_k(t - t_k - \bar{\delta}), \quad (14)$$

for all $t \in [t_k + \bar{\delta}, t_{k+1}]$ and for any $\tilde{d}_k \in \mathcal{C}(\ell_k; I_{\bar{\delta}})$ such that $|\delta - \tilde{d}_k(t)| \le |\delta - d_k(t)|$. Then, the delay estimate $\hat{\delta}(t)$ provided by (7)–(13) is such that $\epsilon(t)$ converges asymptotically to zero.

The proof is given in Section 5.

Remark 5. The main assumption (14) in Theorem 4 is a technical hypothesis to guarantee the convergence to zero of $\epsilon(t)$. However, it is easy to see that if $\|\mathcal{S}_k(t)\|$ is a non-zero function, then (14) is satisfied locally with respect to the delay (i.e. when $|\delta - d_k(t)|$ is sufficiently small). Indeed, since the function $\tilde{d}_k$ is supposed to be such that $|\delta - \tilde{d}_k(t)| \le |\delta - d_k(t)|$, the temporary estimate $d_k(t)$ can be sufficiently close to the real delay δ to make itself sufficiently close to $\tilde{d}_k$ and obtain that $\|\mathcal{S}_k(t)\| > 0$ implies $\mathcal{S}^T_{\tilde{d}_k}(t)\mathcal{S}_k(t) > 0$. Therefore, even if condition (14) cannot be verified *a priori*, we can check *on line* if it is satisfied locally with respect to the delay by verifying if $\|\mathcal{S}_k(t)\|$ is different from zero (see the examples in Section 6).

Next Corollary summarizes the main results this paper by providing the sufficient conditions to obtain the exponential stability of the considered delay system.

Corollary 6. Consider system (1) with the control law (5) and the delay estimate $\hat{\delta}(t)$, provided by (7)–(13). Assume that the hypothesis of Theorem 4 holds true and condition (6) is satisfied for a positive real α. Then, system (1) is exponentially stable.

The proof is given in Section 5.

5. PROOFS

In order to prove Theorem 2 and Theorem 3, we represent system (3) with an *infinite dimensional state space representation*, which is often used for managing delay systems (e.g. Germani et al. (2000); Cacace et al. (2015)). The next subsection introduces this representation. Then, Subsection 5.2 provides the proofs of Theorem 2 and Theorem 3, Subsection 5.3 provides the proof of Theorem 4 and, finally, Corollary 6 is proved in Subsection 5.4.

5.1 Infinite dimensional representation of system (3)

Using the control law (5) and recalling that $\bar{A} = A - B\bar{K}$, system (3) assumes the form

$$\dot{x}(t) = Ax(t) - B\bar{K}e^{\bar{A}\hat{\delta}(t-\delta)}x(t-\delta). \tag{15}$$

Considering that $\epsilon(t) = \delta - \hat{\delta}(t)$, here supposed to be given for $t \geq -\delta$, (15) becomes, for $t \geq 0$,

$$\begin{aligned}\dot{x}(t) = Ax(t) - B\bar{K}e^{\bar{A}\delta}x(t-\delta)\\ + B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})x(t-\delta).\end{aligned} \tag{16}$$

Notice that the last term is null when $\epsilon(t-\delta)$ is equal to zero. Therefore, first two terms coincide to the dynamics of system (3) with $\hat{\delta}(t) = \delta \ \forall t \geq -\delta$.

System (16) can be rewritten in state-space form in the Banach space M_∞, defined as

$$M_\infty := \left\{ \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} : \begin{array}{l} \mathbf{x}_0 \in \mathbb{R}^n \\ \mathbf{x}_1 \in W^{1,2}, \end{array} \quad \mathbf{x}_0 = \mathbf{x}_1(0) \right\}$$

endowed with the norm $\|\mathbf{x}\|_{M_\infty} := \sqrt{\|\mathbf{x}_0\|^2 + \|\mathbf{x}_1\|_\infty^2}$ where $\mathbf{x} \in M_\infty$ is composed by $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{x}_1 \in \mathcal{C}_\delta^n$,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix},$$

and $\|\mathbf{x}_1\|_\infty = \sup_{\theta \in [-\delta, 0]} \|\mathbf{x}_1(\theta)\|^2$.

In M_2, system (16) assumes the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1(t)\mathbf{x}(t), \qquad t \geq 0 \tag{17}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix},$$

is composed by the current state $x(t)$ and $x_t \in W^{1,2}$, which is the state trajectory into the interval $[t-\delta, t]$, i.e. $x_t : x_t(\theta) = x(t+\theta)$ with $\theta \in [-\delta, 0]$. The initial condition is

$$\mathbf{x}(0) = \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}. \tag{18}$$

The operator $\mathbf{A}_0 : \mathcal{D}(\mathbf{A}_0) \to M_2$ is defined as

$$\mathbf{A}_0 : \mathbf{A}_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} A\mathbf{x}_0 - B\bar{K}e^{\bar{A}\delta}\mathbf{x}_1(-\delta) \\ \dfrac{d}{d\theta}\mathbf{x}_1 \end{bmatrix},$$

with domain $\mathcal{D}(\mathbf{A}_0) = M_\infty$. The collection of operators $\mathbf{A}_1(t) : M_\infty \to M_\infty$ is defined for $t \geq 0$ as

$$\mathbf{A}_1(t) : \ \mathbf{A}_1(t) \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})\mathbf{x}_1(-\delta) \\ 0 \end{bmatrix},$$

We can readily prove that for all $t \geq 0$,

$$\|\mathbf{A}_1(t)\|_{M_\infty} \leq \|B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})\| =: \mathbf{F}(t). \tag{19}$$

Indeed, since for any $\mathbf{x} \in M_\infty$, $\|\mathbf{x}_0\|^2 \leq \|\mathbf{x}\|_{M_\infty}^2$, we obtain:

$$\begin{aligned}\|\mathbf{A}_1(t)\|_{M_\infty}^2 &= \sup_{\mathbf{x} \in M_\infty} \frac{\|\mathbf{A}_1(t)\mathbf{x}\|_{M_\infty}^2}{\|\mathbf{x}\|_{M_\infty}^2} \\ &= \sup_{\mathbf{x} \in M_\infty} \frac{\|B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})\mathbf{x}_1(-\delta)\|^2}{\|\mathbf{x}\|_{M_\infty}^2} \\ &\leq \sup_{\mathbf{x} \in M_\infty} \frac{\|B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})\|^2 \|\mathbf{x}_1(-\delta)\|^2}{\|\mathbf{x}\|_{M_\infty}^2} \\ &\leq \|B\bar{K}e^{\bar{A}\delta}(I_n - e^{-\bar{A}\epsilon(t-\delta)})\|^2.\end{aligned}$$

Notice that if $\epsilon(t-\delta) = 0$, then $\mathbf{A}_1(t) = 0$. Moreover, if $|\epsilon(t)|$ is uniformly bounded, then $\mathbf{F}(t)$ is uniformly bounded and $\mathbf{A}(t)$ is a bounded operator. Under this hypothesis, the differential equation (17), with initial condition (18), admits the *mild* solution

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{A}_1(\tau)\mathbf{x}(\tau)d\tau \tag{20}$$

where, for $0 \leq t \leq t_f$ ($t_f < \infty$), $\mathbf{\Phi}(t)$ is a linear bounded operator on M_∞ into itself, strongly continuous in $t \in [0, t_f]$, and such that $\mathbf{\Phi}(t) = \mathbf{\Phi}(t-\theta)\mathbf{\Phi}(\theta)$, $t \geq \theta \geq 0$, and $\mathbf{\Phi}(0)$ is equal to the identity operator in M_∞ (Pazy, 1983).

We also define the operator $\mathbf{\Pi}_n^0 : M_\infty \to \mathbb{R}^n$ as $\mathbf{\Pi}_n^0\mathbf{x} = \mathbf{x}_0$, which simply extracts the first component of the state $\mathbf{x} \in M_\infty$, and we introduce the useful property stated by the following Lemma.

Lemma 7. Consider system (17) with the initial condition (18). Suppose $\epsilon(t) = 0$ for all $t \geq -\delta$, i.e. $\mathbf{A}_1(t) = 0$ for all $t \geq 0$. If there exist $\gamma, \alpha > 0$ such that, for all $t \geq 0$, $\|\mathbf{\Pi}_n^0\mathbf{x}(t)\| \leq \gamma e^{-\alpha t}\|\phi\|_\infty$, with $\phi(\theta) = \mathbf{\Pi}_n^0\mathbf{x}(\theta)$ for all $\theta \in [-\delta, 0]$, then there exists $\Gamma_\alpha > 0$ such that, for all $t \geq 0$, $\|\mathbf{\Phi}(t)\|_{M_\infty} \leq \Gamma_\alpha e^{-\alpha t}$.

Proof. By the definition of M_∞ we have that $\mathbf{x}_t(\theta) = x(t+\theta) = \mathbf{\Pi}_n^0\mathbf{x}(t+\theta)$. Since $\mathbf{A}_1(t) = 0$, from (20) it follows that, for all $t \geq 0$, $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$ and therefore:

$$\begin{aligned}\|\mathbf{\Phi}(t)\|_{M_\infty}^2 &= \sup_{\mathbf{x}(0) \in M_\infty} \frac{\|\mathbf{\Phi}(t)\mathbf{x}(0)\|_{M_\infty}^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} \\ &= \sup_{\mathbf{x}(0) \in M_\infty} \frac{\|\mathbf{x}(t)\|_{M_\infty}^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} \\ &= \sup_{\mathbf{x}(0) \in M_\infty} \frac{\|\mathbf{\Pi}_n^0\mathbf{x}(t)\|^2 + \sup_{\theta \in [-\delta, 0]} \|\mathbf{\Pi}_n^0\mathbf{x}(t+\theta)\|^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} \\ &\leq \sup_{\mathbf{x}(0) \in M_\infty} \frac{\gamma^2 \left(e^{-2\alpha t} + \sup_{\theta \in [-\delta, 0]} e^{-2\alpha(t+\theta)}\right) \|\phi\|_\infty^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} \\ &= \sup_{\mathbf{x}(0) \in M_\infty} \frac{\gamma^2 \left(e^{-2\alpha t} + e^{-2\alpha(t-\delta)}\right) \|\phi\|_\infty^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} \\ &\leq \sup_{\mathbf{x}(0) \in M_\infty} \frac{\gamma^2 \left(1 + e^{2\alpha\delta}\right) e^{-2\alpha t} \|\mathbf{x}(0)\|_{M_\infty}^2}{\|\mathbf{x}(0)\|_{M_\infty}^2} = \Gamma_\alpha^2 e^{-2\alpha t},\end{aligned}$$

with

$$\Gamma_\alpha = \sqrt{\gamma^2(1 + e^{2\alpha\delta})}. \tag{21}$$

5.2 Proof of Theorem 2 and Theorem 3

Even if similar, for clarity of presentation, Theorem 2 and Theorem 3 are given as two different results. Anyway, Theorem 2 actually is a corollary of Theorem 3 since, if $\lim_{t \to +\infty} \epsilon(t) = 0$, the delay identification error $\epsilon(t)$ will surely definitely enter into an arbitrarily small interval around zero. By Theorem 3 this means that the controlled system is exponentially stable. Therefore, in order to prove both the theorems, we need now to only prove Theorem 3.

Proof. Under the hypotheses of Theorem 3, Lemma 1 implies that the delay system (15), with $\hat{\delta}(t) = \delta$ is α-*exp* stable. The state of this system coincides with the component $\mathbf{\Pi}_n^0\mathbf{x}(t)$ of the state of system (17), with

$\mathbf{A}_1(t) = 0$. This implies that the hypotheses of Lemma 7 are satisfied. Therefore, we have that, for all $t \geq 0$, $\|\mathbf{\Phi}(t)\|_{M_\infty} \leq \Gamma_\alpha e^{-\alpha t}$ with Γ_α as in (21). Taking into account such an inequality and the bound for $\|\mathbf{A}_1(t)\|_{M_\infty}$ in (19), from the mild solution (20), we obtain

$$\begin{aligned}
&\|\mathbf{x}(t)\|_{M_\infty} \\
&\leq \|\mathbf{\Phi}(t)\mathbf{x}(0)\|_{M_\infty} + \int_0^t \|\mathbf{\Phi}(t-\tau)\mathbf{A}_1(\tau)\mathbf{x}(\tau)\|_{M_\infty} d\tau \\
&\leq \|\mathbf{\Phi}(t)\|_{M_\infty}\|\mathbf{x}(0)\|_{M_\infty} \\
&\quad + \int_0^t \|\mathbf{\Phi}(t-\tau)\|_{M_\infty}\|\mathbf{A}_1(\tau)\|_{M_\infty}\|\mathbf{x}(\tau)\|_{M_\infty} d\tau \\
&\leq \Gamma_\alpha e^{-\alpha t}\|\mathbf{x}(0)\|_{M_\infty} + \int_0^t \Gamma_\alpha e^{-\alpha(t-\tau)}\mathbf{F}(\tau)\|\mathbf{x}(\tau)\|_{M_\infty} d\tau.
\end{aligned}$$

from which, by using the the integral Bellman-Gronwall inequality, it follows that

$$\|\mathbf{x}(t)\|_{M_\infty} \leq \Gamma_\alpha \|\mathbf{x}(0)\|_{M_\infty} \exp\left(\int_0^t \Gamma_\alpha \mathbf{F}(\tau)d\tau - \alpha t\right). \tag{22}$$

Let us observe now that $\mathbf{F}(t)$ is a function of $\epsilon(t)$ as defined in (19). Since, by hypothesis, $\epsilon(t)$ is uniformly bounded it results that $\mathbf{F}(t)$ is bounded too. Moreover, by continuity, there exists a positive real $\bar{\epsilon}$ such that, if for some $t^* \geq 0$ $|\epsilon(t)| \leq \bar{\epsilon}$, $\forall t \geq t^*$, then $\Gamma_\alpha \mathbf{F}(t) \leq \alpha - \tilde{\alpha}$, $\forall t \geq t^*$, with $0 < \tilde{\alpha} < \alpha$. Thus, from (22) follows that

$$\|\mathbf{x}(t)\|_{M_\infty} \leq \Gamma_\alpha \|\mathbf{x}(0)\|_{M_\infty} e^{T^*} e^{-\tilde{\alpha}t}, \tag{23}$$

where $T^* = \int_0^{t^*} \Gamma_\alpha \mathbf{F}(\tau)d\tau$ is positive and bounded. The proof is concluded by finally noticing that $\|x(t)\| \leq \|\mathbf{x}(t)\|_{M_\infty}$ for all $t \geq 0$.

5.3 Proof of Theorem 4

Proof. Let us define for $t \in \ell_k$ the temporary identification error $\epsilon_k(t)$ as $\epsilon_k(t) = \delta - d_k(t)$. From (11) follows that, for all $t \geq 0$, $\xi_\delta(t) = x(t)$ ($\xi_\delta(t)$ is the version of $\xi_k(t)$ with $d_k(t) = \delta$ for all t and k). Therefore, the residual $r_k(t)$ can be rewritten as $r_k(t) = \xi_\delta(t) - \xi_k(t)$. By the mean value theorem, we can thus write

$$\begin{aligned}
r_k(t) &= \int_{d_k(t)}^{\delta} \frac{\partial \xi_\mu(t)}{\partial \mu} d\mu = \epsilon_k(t) \int_{d_k(t)}^{\delta} \mathcal{S}_\mu(t) d\mu \\
&= \epsilon_k(t)\mathcal{S}_{\tilde{d}_k}(t)
\end{aligned} \tag{24}$$

where $\tilde{d}_k(t)$ is a proper function is $\mathcal{C}(\ell_k; I_{\bar{\delta}})$ such that $|\delta - \tilde{d}_k(t)| \leq |\delta - d_k(t)|$. From (24) and (8), omitting the saturations at 0 and $\bar{\delta}$, we obtain that

$$\begin{aligned}
\dot{\epsilon}_k(t) &= -\frac{\mathcal{S}_{\tilde{d}_k}^T(t)\mathcal{S}_k(t)}{\eta + \beta\|\mathcal{S}_k(t)\|}\epsilon_k(t), \quad t \in [t_k + \bar{\delta}, t_{k+1}], \\
\epsilon_k(t) &= \epsilon_k(t_k), \quad t \in [t_k, t_k + \bar{\delta}],
\end{aligned}$$

which admits the solution, for $t \in [t_k + \bar{\delta}, t_{k+1}]$,

$$\begin{aligned}
\epsilon_k(t) &= \exp\left(-\int_{t_k+\bar{\delta}}^{t} \frac{\mathcal{S}_{\tilde{d}_k}^T(\tau)\mathcal{S}_k(\tau)}{\eta + \beta\|\mathcal{S}_k(\tau)\|} d\tau\right)\epsilon_k(t_k) \\
&= e^{-\mathcal{T}_k(t - t_k - \bar{\delta})}\epsilon_k(t_k),
\end{aligned}$$

from which follows that

$$\epsilon_k(t_{k+1}) = e^{-\mathcal{T}_k(\Delta - \bar{\delta})}\epsilon_k(t_k) = \alpha_k \epsilon_k(t_k). \tag{25}$$

Under the theorem hypothesis $\alpha_k = e^{-\mathcal{T}_k(\Delta-\bar{\delta})} < 1$ for all $k = 0, 1, 2, \ldots$. Therefore, from (25) it follows that $\lim_{k\to\infty} \epsilon_k(t_{k+1}) = 0$. This proves that $\epsilon(t)$ converges asymptotically to zero since by (7) we have $\lim_{t\to+\infty} \epsilon(t) = \lim_{k\to\infty} \epsilon_k(t_{k+1})$.

5.4 Proof of Corollary 6

Proof. Because of the use of the projection operator (9) and recalling that $\delta \leq \bar{\delta}$, we have that $|\epsilon(t)| \leq \bar{\delta}$ for all $t > -\delta$, i.e. it is uniformly bounded. Moreover, Theorem 4 implies that $\lim_{k\to\infty} \epsilon_k(t_{k+1}) = 0$. This means that all the hypotheses of Theorem 2 are satisfied and thus system (1) with the control law (5) is exponentially stable.

6. EXAMPLE

Let us consider the double oscillator system with input delay presented in Cacace et al. (2014) and Yoon and Lin (2013), which has the state space matrices

$$A = \begin{bmatrix} p & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & -\omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{26}$$

For $p = 0.1$ and $\omega = 1$, $\sigma(A) = \{0.1, \ \pm j\}$. The delay bound is supposed to be $\bar{\delta} = 4$. Once $\sigma(\bar{A})$ is set, it is possible to compute the maximal delay $\delta_{\max}$ such that, if δ is known and lower than $\delta_{\max}$, system (1), controlled by (5) is exponentially stable, i.e. condition (6) in Lemma 1 is satisfied for some $\alpha > 0$ (see Cacace et al. (2014) for details). By setting $\sigma(\bar{A}) = \{-0.1, \ -0.1 \pm j\}$ we have, for system (26), $\delta_{\max} = 5.54$ s. Since $\bar{\delta} = 4 < \delta_{\max}$, the hypotheses of Theorems 2 and 3 are satisfied. Consequently, we can obtain the exponential stability of the controlled system by identifying the delay δ using the identifier presented in Section 4.

We consider two cases, both with the true delay value $\delta = 1.5$ s: one with the initial delay estimate $\hat{\delta}_0 = 0$ s (case 1) and one with $\hat{\delta}_0 = 4$ s (case 2). By keeping $\delta(t) = \hat{\delta}_0$ for all $t \geq -\bar{\delta}$, the closed-loop system is unstable in both the cases. Figures 2 and 3 illustrate the obtained results. Figure 2 clearly shows that the closed-loop system is asymptotically stable in both the cases. The state variables converge to zero in about 60 s, when $\hat{\delta}_0 = 0$ s, and in about 80 s, when $\hat{\delta}_0 = 4$ s. Figure 3-(left) depicts the delay estimate, which starts from $\hat{\delta}_0 = 0$ s and increases monotonically toward the actual delay $\delta = 1.5$ s, stopping at about 1.2 after 60 s. We are therefore in the situation of Theorem 3, since the delay identifier does not converge to the real delay but reduces the delay identification error enough for getting the exponential stability of the closed-loop system.

Figure 3 also shows the index ρ_k defined as the ratio between the maxima of the residuals and sensitivities norms $\|r_k(t)\|$ and $\|\mathcal{S}_k(t)\|$, computed over the intervals ℓ_k. When $\hat{\delta}_0 = 0$ s (case 1), these two quantities both tend to zero because of the convergence of the system state. However, since ρ_k converges to a non-zero constant, they have the same decay order. This means that quantities $\|\mathcal{S}_k(t)\|$ are strictly positive. Thus, coherently with Remark 5, condition (14) is locally satisfied for a time

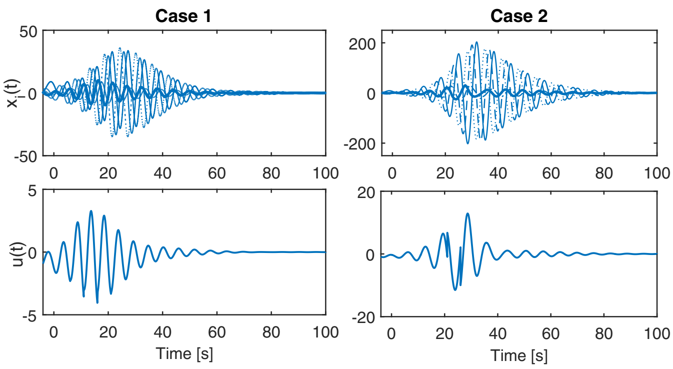

Fig. 2. State and control time evolutions with $\hat{\delta}_0 = 0$ s (left) and $\hat{\delta}_0 = 4$ s (right), $\Delta = 5$ s and $\eta = \beta = 2$.

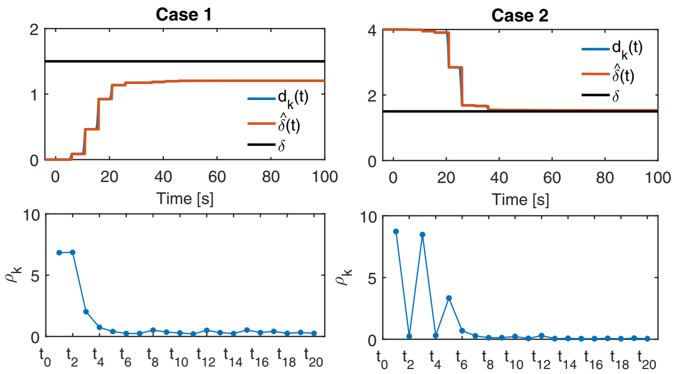

Fig. 3. Delay estimate $\hat{\delta}(t)$ (top) and ratio $\rho_k = \|r_k\|_\infty / \|\mathcal{S}_k\|_\infty$ (bottom), with $\hat{\delta}_0 = 0$ s (left) and $\hat{\delta}_0 = 4$ s (right), $\Delta = 5$ s and $\eta = \beta = 2$.

interval sufficiently large to allow the delay identifier to reduce the delay identification error enough for getting the exponential stability of the closed-loop system.

In Figure 3-(right) we see that, starting from $\hat{\delta}_0 = 4$ s (case 2), the delay identifier converges to the true value. In this case, the ratio ρ_k converges to zero, meaning that $\|\mathcal{S}_k(t)\|$ are always strictly positive. Thus, coherently with Remark 5, condition (14) is locally satisfied allowing the delay identifier to correctly estimate the real delay δ.

7. CONCLUSIONS

In this paper we have proposed a finite-dimensional state feedback able to guarantee the exponential stability of a linear system affected by an unknown constant input delay. The control law uses a delay estimate computed by a delay identifier. We have proved that the convergence of the system state can be obtained when the delay identifier converges to the true value or, more generally, to a neighborhood of it. We also provided a sufficient condition for the convergence or the delay identification error. Further studies are needed to investigate the relationship among the control law, the delay identifiability and the conditions for the convergence of the delay estimate, in order to be able to guarantee *a priori* the effectiveness of the control.

REFERENCES

Belkoura, L. (2005). Identifiability of systems described by convolutionequations. *Automatica*, 41(3), 505–512.

Belkoura, L., Richard, J.P., and Fliess, M. (2009). Parameters estimation of systems with delayed and structured entries. *Automatica*, 45(5), 1117–1125.

Bresch-Pietri, D. and Krstic, M. (2010). Delay-adaptive predictor feedback for systems with unknown long actuator delay. *IEEE Trans. Aut. Contr.*, 55(9), 2106–2112.

Bresch-Pietri, D. and Krstic, M. (2013). *Nonlinear Control under Nonconstant Delays*. SIAM series on Advances in Design and Control.

Bresch-Pietri, D. and Krstic, M. (2014). Delay-adaptive control for nonlinear systems. *IEEE Trans. Autom. Contr.*, 59(5), 1203–1218.

Bresch-Pietri, D., Chauvin, J., and Petit, N. (2012). Adaptive control scheme for uncertain time-delay systems. *Automatica*, 48(8), 1536 – 1552.

Cacace, F., Conte, F., and Germani, A. (2015). Filtering continuous-time linear systems with time-varying measurement delay. *IEEE Trans. Autom. Contr.*, 60(5), 1368–1373.

Cacace, F., Conte, F., Germani, A., and Palombo, G. (2016). Delay identification for a class of nonlinear systems. *International Journal of Control*, 89(11).

Cacace, F., Germani, A., and Manes, C. (2014). Exponential stabilization of linear systems with time-varying delayed statefeedback via partial spectrum assignment. *Systems & Control Letters*, 69, 47–52.

Fridman, E. (2014). *Introduction to Time-Delay Systems*. Birkhäuser, Basel.

Germani, A., Manes, C., and Pepe, P. (2000). A twofold spline approximation for finite horizon LQG control of hereditary systems. *SIAM J. on Control and Optim.*, 39(4), 1233–1295.

Hetel, L., Daafouz, J., Richard, J., and Jungers, M. (2011). Delay-dependent sampled-data control based on delay estimates. *Systems & Control Letters*, 60(2), 146–150.

Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Springer, Heidelberg.

Krstic, M. (2010). Lyapunov stability of linear predictor feedback for time-varying input delay. *IEEE Trans. Autom. Contr.*, 55(2), 554–559.

Manitius, A. and Olbrot, A. (1979). Finite spectrum assignment problem for systems with delays. *IEEE Trans. Autom. Contr.*, 24(4), 541–552.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer.

Polyakov, A., Efimov, D., Perruquetti, W., and Richard, J.P. (2013). Output stabilization of time-varying input delay systems using interval observation technique. *Automatica*, 49(11), 3402–3410.

W. Michiels, T.V. and Ztek, P. (2010). Control design for time-delay systems based on quasi-direct pole placement. *Journal of Process Control*, 20, 337–343.

Yoon, S.Y. and Lin, Z. (2013). Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay. *Systems & Control Letters*, 62(10), 837–844.

Zhou, B. (2014). Pseudo-predictor feedback stabilization of linear systems with time-varying input delays. *Automatica*, 50(11), 2861–2871.

Zhou, B., Lin, Z., and Duan, G.R. (2012). Truncated predictor feedback for linear systems with long time-varying input delays. *Automatica*, 48(10), 2387–2399.