Short-term interest rate estimation by filtering in a model linking inflation, Central Bank and short-term interest rates

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Abstract

We consider the model of Antonacci, Costantini, D'Ippoliti, Papi (arXiv:2010.05462 [q-fin.MF], 2020), which describes the joint evolution of inflation, the Central Bank interest rate and the short-term interest rate. In the case when the diffusion coefficient does not depend on the Central Bank interest rate, we derive a semiclosed valuation formula for contingent derivatives, in particular for Zero Coupon Bonds (ZCB). By using ZCB yields as observations, we implement the Kalman filter and we obtain a dynamical estimate of the short-term interest rate. In turn, by this estimate, at each time step we calibrate the model parameters under the risk neutral measure and the coefficient of the risk premium. We compare the market values of German interest rate yields for several maturities with the corresponding values predicted by our model, from 2007 to 2015. The numerical results validate both our model and our numerical procedure.

Keywords: Inflation, short-term interest rate, Kalman filter, calibration

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1 Introduction

There is empirical and theoretical evidence that bond prices, inflation, interest rates, monetary policy and output growth are related. In particular, both inflation and interest rates are clearly related to the activity of central banks. See Akram and Li [AL20] for a recent discussion of the role of interest rates. In their seminal work of 2003, Jarrow and Yildirim [JY03] proposed an approach based on foreign currency and interest rate derivatives valuation. Singor et al. [SGVBO13] formulated a Heston-type inflation model in combination with a Hull-White model for interest rates, with non-zero correlations. More recently, D'Amico, Kim and Wei [DKW18], Ho, Huang and Yildirim [HHY14] and Waldenberger [W17]) considered affine models with hidden stochastic factors. Discrete time models have also been proposed: Hughston and Macrina [HM08] propose a discrete time model based on utility functions; Haubric et al. [HPR12] develop a discrete time model of nominal and real bond yield curves based on several stochastic drivers.

In the present work we start from a model for the joint evolution of the inflation rate, the Central Bank official interest rate and the short-term interest rate that we have proposed in a recent paper ([ACDP20]): To the best of our knowledge, it is the first model that takes into account the interaction among all these three macroeconomic factors. In our model,

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under the risk neutral probability measure, the inflation rate is modeled as a piecewise constant process that jumps at fixed times t_i; the new value at t_i is given by a Gaussian random variable with expectation depending on the previous value of the inflation rate and on the current value of the Central Bank interest rate. The Central Bank (henceforth CB) official interest rate evolves as a pure jump process with jump intensity and distribution that depend both on its current value and on the current value of inflation. Finally, the short-term interest rate follows a CIR type model with reversion towards an affine function of the CB interest rate and diffusion coefficient depending on the spread between itself and the CB interest rate. For this model, in [ACDP20] we derived the valuation equation for a general European type contingent claim.

While inflation and the CB interest rate are directly observable market factors, the short-term interest rate is not. On the other hand the short-term interest rate is an essential component of all bond and derivative prices. In this work we apply filtering techniques to the [ACDP20] model to estimate dynamically the evolution of the short-term interest rate and at the same time calibrate dynamically the model parameters, from a panel of bond yields. The estimated values can then be used to price other securities. As an example, we use our estimates to compute the ZCB 20-year yield over an eight year period of time, and compare it with market data.

More precisely we consider the [ACDP20] model in the case when the short-term interest rate evolution depends on the CB interest rate only through the drift coefficient (Section 2). Analogously to what is usually done for the CIR model, we suppose that the risk premium is linear: Then our model mantains the same structure under the historical probability measure as under the risk neutral one (see Subsection 2.2). We obtain a semiclosed formula for the solution of the valuation equation (see Section 3). Applied to a Zero Coupon Bond (ZCB in the sequel) our formula yields an equation linking the ZCB yield to the short-term interest rate: This allows to use ZCB yields as observations, while the short-term interest rate represents the state. We discretize the equations in time and approximate them in such a way that we can use the Kalman filter (Section 4). In the past, the Kalman filter has been widely applied to term-structure models: for instance, Babbs and Nowman in [BN99] use it for a generalized Vasicek model; moreover Chatterjee [C05], Chen and Scott [CS03] and Long Vo [V14] use it for the CIR model, etc. Christoffersen and others [CDJKS14] use instead the unscented Kalman filter and a particle filter to try to capture nonlinearities in affine models.

Of course the observation equation can also be used to make a one-step-ahead prediction of the value of the ZCB yield from the filter estimated value of the short-term interest rate. At each time, we use the value of the ZCB yield predicted in this way and the observed value to make a quasi-maximum likelihood estimation of the model parameters. Note that we estimate both the parameters under the risk neutral measure and the coefficient of the risk premium.

We implement our filter on market data for inflation, the European CB interest rate and German bonds with several maturities, for the period March 2007 to December 2015 (details on the numerical procedure are contained in the Appendix). Then we compare the market yield values with the corresponding values predicted by our model (Section5). The model implied values follow the market values quite closely, even though interest rates have undergone major changes during the considered period. Moreover we use the values of the short-term interest rate and of the model parameters obtained by our procedure from the maturities up to 10 years, to predict the 20-year yield over the same period of time: Our model performs quite well in this out-of-sample forecast as well (see Figure 7).

Our model is not a "black box" one, but attempts to capture the interactions among inflation, the CB interest rate and the short-term interest rate employing relatively few parameters. Table 2 reports the values of the parameters calibrated using the information from the first 100 months. In particular the value of the coefficient that in our model links the short-term interest rate to the CB interest rate confirms that the interaction between these two factors cannot be neglected.

All the above results open up several perspectives. In particular our estimate of the short-term interest rate can be used to evaluate more sophisticated fixed-income derivative instruments (for example credit risk derivatives): We intend to pursue this in the future.

2 The model

2.1 The model under a risk neutral probability measure

We suppose that, under a risk neutral probability measure, the triple (Π, R, R^{sh}) of the inflation rate, the CB interest rate and the short-term interest rate follows the model described in this section.

With the usual convention that one year is an interval of length one, let $\mathcal{T} := \{t_i\}_{i\geq 0,\dots,M}$ be the sequence of times at which the values of the inflation rate process $\{\Pi(t_i)\}_{t_i\in\mathcal{T}}$ are observed, where $t_0 = 0$, $t_1 = \frac{1}{12}$ and, for $i \geq 2$, $t_i = it_1$. The evolution is then given by

$$\begin{cases} \Pi(0) = \Pi_0, & \\ \Pi(t) = \Pi(t_i), & t_i \leq t < t_{i+1}, \\ \Pi(t_{i+1}) = \beta\left(\Pi(t_i), R(t_{i+1}^-)\right) + \eta_{i+1}, & t = t_{i+1}, \end{cases} \tag{2.1}$$

where β is an affine function defined by

$$\beta(\pi, r) := \beta_0\pi + k^{\Pi}(\pi^* - \pi) + \beta_{\,1} r = (\beta_0 - k^{\Pi})\pi + k^{\Pi}\pi^* + \beta\, r, \tag{2.2}$$

with $\beta_0,\ \beta_1 \in \mathbb{R}$ and $k^{\Pi},\ \pi^* \in \mathbb{R}_+$ constant parameters such that $0 < \beta_0 - k^{\Pi} < 1$. The fluctuations $\{\eta_i\}_{i=1,\dots,M}$ are i.i.d. random variables distributed according to the $\mathcal{N}(0, v^2)$ law.

As for the CB interest rate, R, it is the solution of the following stochastic equation

$$R(t) = R_0 + \int_0^t J\left(\Pi(s^-), R(s^-), U_{N(s^-)+1}\right) dN(s), \tag{2.3}$$

where $N = N(t)$ is a Poisson process with intensity λ, $\{U_n\}_{n\geq 0}$ are i.i.d. $[0,1]$-uniform random variables, independent of N, and

$$\begin{array}{rcl} J(\pi, r, u) & := & -m\delta\, \mathbf{1}_{(0,1]}(q(\pi, r, -m\delta))\mathbf{1}_{[0, q(\pi, r, -m\delta)]}(u) \\ & & + \sum_{k=-m+1}^{m} k\delta\ \mathbf{1}_{(0,1]}(q(\pi, r, k\delta))\, \mathbf{1}_{(\sum_{h=-m}^{k-1} q(\pi, r, h\delta), \sum_{h=-m}^{k} q(\pi, r, h\delta)]}(u), \end{array} \quad u \in [0,1], \tag{2.4}$$

where $q(\pi, r, k\delta)$ is the probability of a size $k\delta$ jump ($q(\pi, r, k\delta) \geq 0$, $\sum_{k=-m}^{m} q(\pi, r, k\delta) = 1$ for each (π, r)). We assume that

$$q(\cdot, \cdot, k\delta) \text{ are continuous on } \mathbb{R} \times [\underline{r}, \overline{r}] \tag{2.5}$$

$$\begin{array}{c} q(\pi, \cdot, k\delta) \text{ has a finite left derivative at } \overline{r} - k\delta \\ q(\pi, \cdot, -k\delta) \text{ has a finite right derivative at } \underline{r} + k\delta \end{array} \tag{2.6}$$

and that

$$q(\pi, r, k\delta) = 0, \qquad \text{for } r + k\delta \notin (\underline{r}, \overline{r}), \tag{2.7}$$

so that $R(t)$ stays in $(\underline{r}, \overline{r})$ for all times. Of course we can suppose, without loss of generality, $\underline{r} \leq 0$ and $\overline{r} > 0$.

Finally, the evolution of the short-term interest rate, R^{sh}, is a mean-reverting Ito process with coefficients depending on the CB interest rate, R, and hence indirectly on the inflation rate Π as well. More precisely, we suppose that R^{sh} satisfies the following equation

$$\begin{cases} dR^{sh}(t) = k^{sh}\big(b(R(t)) - R^{sh}(t)\big)dt + \sigma_0\sqrt{|R^{sh}(t)|}dW(t), \\ R^{sh}(0) = R_0^{sh}, \end{cases} \tag{2.8}$$

where $k^{sh} \in \mathbb{R}_+$ is a constant parameter and $\{W_t\}_{t\in[0,T]}$ is a standard Wiener process. The function $b(r)$ is defined by

$$b(r) := b_0 + b_1 r, \tag{2.9}$$

where $b_0, b_1 \in \mathbb{R}$ are constant parameters and $\inf_{(\underline{r},\overline{r})} b(r) > 0$. In analogy to the CIR model, we assume that

$$k^{sh} \inf_{(\underline{r},\overline{r})} b(r) > \frac{1}{2}\sigma_0^2. \tag{2.10}$$

The stochastic factors $\{\eta_i\}_{i=1,\dots,M}$, $\{U_n\}_{n\geq 0}$, N, W and the initial values Π_0, R_0, R_0^{sh} are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (where $\mathbb{P}$ is a risk neutral probability measure) and are independent.

Our model is well posed, in the sense that there exists one and only one stochastic process $\left(\Pi, R, R^{sh}\right)$ verifying (2.1), (2.3) and (2.8), as stated precisely in the following theorem, which is proved in [ACDP20] for a more general model that includes the above one.

Theorem 2.1 *([ACDP20], Theorem 2.1) For every triple of $\mathbb{R}\times(\underline{r},\overline{r})\times(0,+\infty)$-valued r.v.'s $\left(\Pi_0, R_0, R_0^{sh}\right)$, there exists one and only one (up to indistinguishibility) stochastic process (Π, R, R^{sh}) defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that (2.1), (2.3) and (2.8) are $\mathbb{P}$-a.s. verified. It holds*

$$R^{sh}(t) > 0 \qquad \forall t \geq 0, \quad \mathbb{P} - a.s..$$

2.2 The model under the historical probability measure

In this section we discuss the structure of our model under the historical probability measure.

As in the previous section, $\mathbb{P}$ denote a risk neutral probability measure on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$. We denote by $\epsilon_{u_0}(\cdot)$ the Dirac measure on u_0.

First of all, we rewrite equations (2.1) and (2.3) with the formalism of random measures (see, e.g. [JS03], Chapter II, Section 1b.).

(2.1) can be rewritten as

$$\begin{aligned}
\Pi(t) &= \Pi_0 + \int_{0^+}^{t^+}\int_{\mathbb{R}} u\; \mu^{\Pi}(du, ds), \\
\mu^{\Pi}(ds, du) :&= \epsilon_{\Delta\Pi(s)}(du)\; dI(s), \\
I(s) :&= \sum_{i=1}^{\infty} \mathbf{1}_{[0,s]}(t_i).
\end{aligned} \tag{2.11}$$

Analogously (2.3) can be rewritten as

$$\begin{aligned}
R(t) &= R_0 + \int_{0^+}^{t^+}\int_{(-m\delta, m\delta)} u\; \mu(du, ds), \\
\mu(ds, du) :&= \epsilon_{\Delta R(s)}(du)\; dN(s),
\end{aligned} \tag{2.12}$$

Theorem 2.2 *i) The compensators of μ^{Π} and μ are, respectively,*

$$\nu^{\Pi}(ds, du) := G(\beta\left(\Pi(s^-), R(s^-)\right) - \Pi(s^-), u)\, du\, ds \tag{2.13}$$

$$G(\beta(\pi, r) - \pi, u) := \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(u - \beta(\pi,r) + \pi\right)^2}{2v^2}},$$

$$\nu(ds, du) := \Lambda(s, du)\, \lambda\, ds, \qquad (2.14)$$

$$\sum_{k=-m}^{m} q(\Pi(s^-), R(s^-), k\delta)\, \epsilon_{k\delta}(du)$$

ii) (Girsanov's Theorem) Let $\mathcal{F}_t := \sigma\big(\Pi(s), R(s), R^{sh}(s),\, s \le t\big)$ *and let* $\overline{\mathbb{P}}$ *denote a probability measure on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ *locally equivalent to* $\mathbb{P}$ *(i.e.* $\overline{\mathbb{P}}\big|_{\mathcal{F}_t}$ *is equivalent to* $\mathbb{P}\big|_{\mathcal{F}_t}$ *for every* $t \ge 0$*). Then there exist a predictable process* $\gamma = \gamma(t)$*,* $\Gamma : \Omega \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$*,* $\mathcal{P} \times \mathcal{B}([0,1])$*-measurable,* $\Gamma^\pi : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$*,* $\mathcal{P} \times \mathcal{B}(\mathbb{R})$*-measurable (where* $\mathcal{P}$ *is the predictable* σ*-algebra and* $\mathcal{B}$ *denotes the Borel* σ*-algebra) such that, under* $\overline{\mathbb{P}}$*:*

a) $W(t) - \int_0^t \gamma(s) ds$ *is a Brownian motion;*

b) the compensator of μ^Π *is* $\overline{\nu}^\Pi(ds, du) := \Gamma^\Pi(s, u)\, \nu^\Pi(ds, du)$*;*

c) the compensator of μ *is* $\overline{\nu}(ds, du) := \Gamma(s, u)\, \nu(ds, du)$*,*

where ν^Π *and* ν *are defined by* (2.13) (2.14)

iii) In particular, under $\overline{\mathbb{P}}$ *the process* R^{sh} *satisfies the equation*

$$\begin{cases} dR^{sh}(t) = \left[k^{sh}\big(b(R(t)) - R^{sh}(t)\big)\big) + \sigma_0 \sqrt{R^{sh}(t)}\gamma(t)\right] dt + \sigma_0 \sqrt{R^{sh}(t)} d\overline{W}(t), \\ R^{sh}(0) = R_0^{sh}, \end{cases} \qquad (2.15)$$

where $\overline{W}$ *is an* $\{\mathcal{F}_t\}$*-Brownian motion under* $\overline{\mathbb{P}}$*.*

Proof.
i) It is enough to show that, for every $\xi : [0, \infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$, $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable and bounded, with ν^Π and ν defined by (2.13) and (2.14),

$$M^\Pi(t) := \int_{0^+}^{t^+} \int_{\mathbb{R}} \xi(s, u) \left(\mu^\Pi(ds, du) - \nu^\Pi(ds, du)\right),$$

$$M(t) := \int_{0^+}^{t^+} \int_{(-m\delta, m\delta)} \xi(s, u) \left(\mu(ds, du) - \nu(ds, du)\right),$$

are $\{\mathcal{F}_t\}$-martingales. We have

$$\begin{aligned} & \mathbb{E}\left[M^\Pi(t) - M^\Pi(s) \middle| \mathcal{F}_s\right] \\ = \;& \mathbb{E}\left[\int_{s^+}^{t^+} \left(\xi(l, \Delta\Pi(l)) - \int_{\mathbb{R}} \xi(l, u)\, G(\Pi(l^-), R(l^-), u)\, du\right) dI(l) \middle| \mathcal{F}_s\right] \\ = \;& \mathbb{E}\left[\sum_{s < t_i \le t} \left(\xi(t_i, \Delta\Pi(t_i)) - \int_{\mathbb{R}} \xi(t_i, u)\, G(\Pi(t_i^-), R(t_i^-), u)\, du\right) \middle| \mathcal{F}_s\right] \\ = \;& \sum_{s < t_i \le t} \mathbb{E}\left[\mathbb{E}\left[\left(\xi\big(t_i, \Delta\Pi(t_i)\big) - \int_{\mathbb{R}} \xi(t_i, u)\, G(\Pi(t_i^-), R(t_i^-), u)\, du\right) \middle| \mathcal{F}_{t_i^-}\right] \middle| \mathcal{F}_s\right] \\ = \;& 0 \end{aligned}$$

where the last equality follows from the fact that, by (2.1), the law of $\Delta\Pi(t_i)$ conditionally to $\mathcal{F}_{t_i^-}$ is $G(\Pi(t_i^-), R(t_i^-), u)\, du$.

Analogously

$$
\begin{aligned}
& \mathbb{E}\big[M(t) - M(s)\big|\mathcal{F}_s\big] \\
= \;& \mathbb{E}\bigg[\int_{s^+}^{t^+} \xi(l, \Delta R(l))\, dN(l) \; - \lambda \int_s^t \int_{(-m\delta, m\delta)} \xi(l, u)\, \Lambda(l, du)\, dl \bigg| \mathcal{F}_s\bigg] \\
= \;& \mathbb{E}\bigg[\int_{s^+}^{t^+} \Big(\xi(l, \Delta R(l)) - \int_{(-m\delta, m\delta)} \xi(l, u)\, \Lambda(l, du)\Big) dN(l) \; + \int_{s^+}^{t^+} \int_{(-m\delta, m\delta)} \xi(l, u)\, \Lambda(l, du)\, (dN(l) - \lambda dl) \bigg| \mathcal{F}_s\bigg] \\
= \;& \mathbb{E}\bigg[\int_{s^+}^{t^+} \Big(\xi(l, \Delta R(l)) - \int_{(-m\delta, m\delta)} \xi(l, u)\, \Lambda(l, du)\Big) dN(l) \bigg| \mathcal{F}_s\bigg],
\end{aligned}
$$

where the last equality follows from the fact that the second integral in the last but one line vanishes because $N(t) - \lambda t$ is a martingale and the integrand is predictable. Our chain of equalities can be continued as

$$
= \; \mathbb{E}\bigg[\int_{s^+}^{t^+} \mathbb{E}\bigg[\xi(l, \Delta R(l)) - \int_{(-m\delta, m\delta)} \xi(l, u)\, \Lambda(l, du) \bigg| \mathcal{F}_{l^-}\bigg] dN(l) \bigg| \mathcal{F}_s\bigg]
$$

and the inside expectations vanish because, by (2.3), $\Lambda(\tau_i, du)$ is the law of $\Delta R(l)$ conditionally on $\mathcal{F}_{l^-}$.

ii) The assertion is a direct consequence of Theorem 3.24, Chapter III, of [JS03].

iii) Setting

$$
\overline{W}(t) := W(t) - \int_0^t \gamma(s)\, ds,
$$

R^{sh} verifies (2.15). □

In analogy to the CIR model, in the sequel we will assume the following.

Assumption 2.3 *For any risk neutral probability measure, the risk premium is linear in R^{sh}, that is, for the historical probability measure the process γ (i.e. the market price of risk) is of the form*

$$
\gamma(t) := \frac{\vartheta}{\sigma_0} \sqrt{R^{sh}(t)}. \tag{2.16}
$$

With Assumption 2.3, under the historical probability measure $\overline{\mathbb{P}}$, (2.15) can be rewritten as

$$
\begin{cases} dR^{sh}(t) = \big[k^{sh}\big(b(R(t)) - R^{sh}(t)\big)\big) + \vartheta R^{sh}(t)\big] dt + \sigma_0 \sqrt{R^{sh}(t)} d\overline{W}(t), \\ R^{sh}(0) = R^{sh}_0. \end{cases} \tag{2.17}
$$

3 A semiclosed valuation formula

In [ACDP20], for a more general model that includes the one described in Section 2, we derived the price of a European type contingent claim with maturity

$$
t_M \le T < t_{M+1}, \tag{3.1}
$$

payoff

$$
V(T)^p\, \Phi(\Pi(T), R(T), R^{sh}(T)), \quad p \ge 0, \tag{3.2}
$$

where V is the inflation index, i.e.

$$
V(t) := \exp\Big(\int_0^t \Pi(l) dl\Big). \tag{3.3}
$$

Theorem 3.1 *([ACDP20], Proposition 3.4 and Theorem 3.5)*

The price at time s of the contingent claim with payoff (3.2) *is*

$$V(s)^p\varphi(s,\Pi(s),R(s),R^{sh}(s)),$$

where

$$\varphi(s,\pi,r,z)) := \begin{cases} \mathrm{e}^{p(T-s)\pi}\,\varphi^M(s-t_M,\pi,r,z), & t_M \le s \le T, \\ \mathrm{e}^{p(t_{i+1}-s)\pi}\varphi^i(s-t_i,\pi,r,z), & t_i \le s < t_{i+1},\ i=0,\dots,M-1, \end{cases}$$

and φ^i is the solution of the equation

$$\begin{aligned} &\tfrac{\partial \varphi^i}{\partial s}(s,r,z) + k^{sh}\,(b(r)-z)\,\tfrac{\partial \varphi^i}{\partial z}(s,r,z) + \tfrac{1}{2}\sigma_0^2 z\tfrac{\partial^2\varphi^i}{\partial z^2}(s,r,z) \\ &+\lambda \textstyle\sum_{k=-m}^{m}\left[\varphi^i\left(s,r+k\delta,z\right)-\varphi^i\left(s,r,z\right)\right]q(\pi,r,k\delta) - z\varphi^i(s,r,z)=0, \end{aligned} \tag{3.4}$$

with terminal condition

$$\begin{aligned} \varphi^M(T-t_M,\pi,r,z) &= \Phi(\pi,r,z), && \text{for } \varphi^M, \\ \varphi^{M-1}(t_1,\pi,r,z) &= B\left(e^{p(T-t_M)\cdot}\varphi^M(0,\cdot,r,z)\right)(\pi,r), && \text{for } \varphi^{M-1}, \\ \varphi^i(t_1,\pi,r,z) &= B\left(e^{pt_1\cdot}\varphi^{i+1}(0,\cdot,r,z)\right)(\pi,r), && \text{for } \varphi^i,\ \ i=0,...,M-2. \end{aligned} \tag{3.5}$$

$$Bf(\pi,r) := \mathbb{E}[f(\beta(\pi,r)+\eta)], \quad \eta \text{ a } \mathcal{N}(0,v^2) \text{ random variable..} \tag{3.6}$$

Remark 3.2 *Allowing payoffs of the form* (3.2)*, in particular we can price a contingent claim with payoff Φ under the discount factor* $\exp\left(-\int_s^T\left(R^{sh}(l)-\Pi(l)\right)dl\right)$*. In fact, in this case, the price of the derivative at time s is*

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_s^T\left(R^{sh}(l)-\Pi(l)\right)dl\right)\Phi(\Pi(T),R(T),R^{sh}(T))\middle|\mathcal{F}_s\right] \\ =\ & \frac{1}{V(s)}\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_s^T R^{sh}(l)dl\right)V(T)\Phi(\Pi(T),R(T),R^{sh}(T))\middle|\mathcal{F}_s\right]. \\ =\ & \varphi(s,\Pi(s),R(s),R^{sh}(s)). \end{aligned} \tag{3.7}$$

Note that the value of function φ does not depend upon the current value of V, but the function itself changes with the value of the parameter p, in this case $p=1$.

In the following Proposition we show that, for payoffs of a suitable form, the solution of the valuation equation (3.4) admits a semiclosed expression.

Proposition 3.3 *Suppose Φ is of the form*

$$\Phi(\pi,r,z) = \Phi_0(\pi,r)\mathrm{e}^{\alpha_0 z}.$$

Then the price of the contingent claim with payoff (3.2) *at time s is*

$$V(s)^p\varphi(s,\Pi(s),R(s),R^{sh}(s)),$$

where

$$\varphi(s,\pi,r,z) = \varphi_0(s,\pi,r)\mathrm{e}^{\alpha(s)z}, \tag{3.8}$$

with

$$\alpha(s) = -2\frac{(1+\alpha_0\rho_-) - (1-\alpha_0\rho_+)\mathrm{e}^{-(\rho_+-\rho_-)(T-s)}}{(2\rho_+ - \alpha_0\sigma_0^2) - (2\rho_- - \alpha_0\sigma_0^2)\mathrm{e}^{-(\rho_+-\rho_-)(T-s)}}, \tag{3.9}$$
$$\rho_+ = \frac{k^{sh} + \sqrt{(k^{sh})^2 + 2\sigma_0^2}}{2}, \quad \rho_- = \frac{k^{sh} - \sqrt{(k^{sh})^2 + 2\sigma_0^2}}{2},$$

and

$$\varphi_0(s,\pi,r) = \begin{cases} \mathrm{e}^{p(T-s)\pi}\,\varphi_0^M(s-t_M,\pi,r), & t_M \le s \le T, \\ \mathrm{e}^{p(t_{i+1}-s)\pi}\varphi_0^i(s-t_i,\pi,r), & t_i \le s < t_{i+1},\ i=0,\dots,M-1. \end{cases} \tag{3.10}$$

the φ_0^i's being solutions of the equation

$$\begin{aligned} &\tfrac{\partial\varphi_0^i}{\partial s}(s,r) + \left(k^{sh}b(r)\alpha(t_i+s)\right)\varphi_0^i(s,r) \\ &+\bar{\lambda}\textstyle\sum_{k=-m}^{m}\left[\varphi_0^i\left(s,r+k\delta\right) - \varphi_0^i\left(s,r\right)\right]q(\pi,r,k\delta) = 0, \end{aligned} \tag{3.11}$$

with terminal conditions

$$\begin{aligned} &\varphi_0^M(T-t_M,\pi,r) = \Phi_0(\pi,r), \\ &\varphi_0^{M-1}(t_1,\pi,r) = B\left(e^{p(T-t_M)\cdot}\varphi_0^M(0,\cdot,r)\right)(\pi,r), \\ &\varphi_0^i(t_1,\pi,r) = B\left(e^{pt_1\cdot}\varphi_0^{i+1}(0,\cdot,r)\right)(\pi,r), \text{for } i=0,...,M-2. \end{aligned} \tag{3.12}$$

Proof. We look for a solution of the valuation equations (3.4) of the form $\varphi^i(s,\pi,r,z) = \varphi_0^i(s,\pi,r)\mathrm{e}^{\alpha_i(s)z}$. By substituting in (3.4), we find that a function of the above form is a solution if the α_i's satisfy the Riccati equations on $[0,t_1]$ ($[0,T-t_M]$ for $i=M$)

$$\alpha_i'(s) - k^{sh}\alpha_i(s) + \frac{1}{2}\sigma_0^2\alpha_i(s)^2 - 1 = 0, \quad \alpha_i(t_1) = \alpha_{i+1}(0), \quad i=0,...,M-1, \quad \alpha_M(T-t_M) = \alpha_0,$$

and the φ_0^i's satisfy (3.11). Therefore we can take

$$\alpha_i(s) = \alpha(t_i+s),$$

where α is the solution of the Riccati equation on $[0,T]$

$$\alpha'(s) - k^{sh}\alpha(s) + \frac{1}{2}\sigma_0^2\alpha(s)^2 - 1 = 0, \quad \alpha(T) = \alpha_0, \tag{3.13}$$

and hence is given by (3.10). □

Remark 3.4 *From* (3.8) *we observe that the contingent claim price factors into two components: one depending only on inflation and the CB interest rate, and the other one depending only on the short-term interest rate. In particular the ZCB yield is linear in R^{sh} and this allows to employ the Kalman filter in the next section.*

4 Short-term interest rate estimation by filtering

In our model, one can obtain a dynamical estimate of the short-term interest rate $R^{sh}(t)$ by filtering, using ZCB yields as observations.

As inflation is observed monthly, we consider $t_i = \frac{i}{12}$, $i \in \mathbb{Z}_+$, the time unit being 1 year. Let $Y_j(t_i)$ be the market value of the yield of a ZCB with time to maturity T_j, $j = 1, ..., J$, and $Y(t_i)$ be the vector of components $Y_j(t_i)$. By Proposition 3.3 (with $p = 0$), the theoretical value of the ZCBs' yields is given by

$$
\begin{aligned}
& -\frac{1}{T_j} \log\left(\varphi_j^{t_i}(t_i, \Pi(t_i), R(t_i), R^{sh}(t_i))\right) \\
= \quad & -\frac{1}{T_j} \log\left(\varphi_{0,j}^{t_i}(t, \Pi(t_i), R(t_i)) \mathrm{e}^{\alpha_j^{t_i}(t_i) R^{sh}(t_i))}\right) \\
= \quad & -\frac{1}{T_j} \log\left(\varphi_{0,j}^{t_i}(t_i, \Pi(t_i), R(t_i)) \mathrm{e}^{\alpha_j^{0}(0) R^{sh}(t_i))}\right) \\
= \quad & -\frac{1}{T_j} \log\left(\varphi_{0,j}^{t_i}(t, \Pi(t_i), R(t_i))\right) - \frac{1}{T_j} \alpha_j^0(0) R^{sh}(t_i)),
\end{aligned}
\tag{4.1}
$$

where the superscript t_i reminds that the maturity of the bond is $t_i + T_j$. Note that the model is not time homogeneous, that is $\varphi_{0,j}^t(t, \pi, r, z) \neq \varphi_{0,j}^0(0, \pi, r, z)$, due to the inflation component. However $\alpha_j^t(t) = \alpha_j^0(0)$ because the equation (3.13) is time homogeneous. Denoting by h_0 the J-dimensional vector of components

$$
(h_0)_j := -\frac{1}{T_j} \alpha_j^0(0), \tag{4.2}
$$

and by $\psi(t_i)$ the vector of components

$$
\psi_j(t_i) := -\frac{1}{T_j} \log\left(\varphi_{0,j}^{t_i}(t_i, \Pi(t_i), R(t_i))\right), \tag{4.3}
$$

the theoretical value of the yield vector at time t_i is

$$
R^{sh}(t_i)\, h_0 + \psi(t_i).
$$

We suppose that the inflation value, Π, and the CB interest rate, R, are perfectly observable, while the ZCB yields are observed with some error, that is

$$
Y(t_i) = R^{sh}(t_i)\, h_0 + \psi(t_i) + \zeta_i. \tag{4.4}
$$

$R^{sh}(t_i)$ is not observable and its dynamics, under the historical probability measure, is given by the time-discretized version of (2.17):

$$
R^{sh}(t_i) = \left(1 - \overline{k}^{sh}(t_i - t_{i-1})\right) R^{sh}(t_{i-1}) + k^{sh}(t_i - t_{i-1})\, b(R(t_{i-1})) + \sigma_0 \sqrt{R^{sh}(t_{i-1})}\, \overline{W}_i, \tag{4.5}
$$

where

$$
\overline{k}^{sh} := k^{sh} - \vartheta
$$

$\vartheta R^{sh}(t_{i-1})$ is the risk premium and $\{\overline{W}_i\}$ are independent, Gaussian random variables, independent of $R^{sh}(0)$, with zero mean and variance $(t_i - t_{i-1})$. The observation errors $\{\zeta_i\}$ are independent, J-dimensional, Gaussian random {variables with covariance matrix a diagonal matrix Q, and the sequence $\{\zeta_i\}$ is independent of $\{\overline{W}_i\}$ and $R^{sh}(0)$, and hence of $\{R^{sh}(t_i\})$.

At each time t_i the estimate, $\widehat{R}^{sh}(t_i)$, of $R^{sh}(t_i)$ is obtained by a suitable filter of equations (4.5) – (4.4) (Subsection 4.1).

In addition, we recalibrate the parameters of the model at each time t_i by a quasi-maximum likelihood estimation based on the errors $\{Y(t_h) - \widehat{Y}(t_h),\, h \leq i\}$, where $\widehat{Y}(t_h)$ is a prediction of $Y(t_h)$ obtained from $\widehat{R}^{sh}(t_{h-1})$ and $R(t_h)$ (Subsection 4.2).

In Section 5, we validate our estimates $\{\widehat{R}^{sh}(t_i)\}$, by comparing the values $\{\widehat{Y}(t_i)\}$ predicted by the model with the observed market values $\{Y(t_i)\}$.

4.1 The filter

Consider equations (4.5) – (4.4). In the filtering terminology (4.5) is the state equation and (4.4) is the observation equation. In equation (4.5), $\sigma_0\sqrt{R^{sh}(t_{i-1})}\overline{W}_i$, conditionally on $\mathcal{F}^Y_{t_{i-1}} \vee \sigma(R^{sh}(t_{i-1}))$, follows a Gaussian law with zero mean and variance

$$\sigma_0^2 R^{sh}(t_{i-1})(t_i - t_{i-1})$$

because $\overline{W}_i$ is independent of $R^{sh}(t_{i-1})$. We replace $\{\sigma_0\sqrt{R^{sh}(t_{i-1})}\overline{W}_i\}$ by a sequence of random variables $\{\xi_i\}$, with ξ_i that follows, conditionally on $\mathcal{F}^Y_{t_{i-1}} \vee \sigma(R^{sh}(t_{i-1}))$, a Gaussian law with zero mean and variance

$$\sigma_0^2 \mathbb{E}\big[R^{sh}(t_{i-1})\big|\mathcal{F}^Y_{t_{i-1}}\big](t_i - t_{i-1}). \tag{4.6}$$

Then our state and observation equations at time t_i are

$$\begin{aligned} R^{sh}(t_i) &= \big(1 - \overline{k}^{sh}(t_i - t_{i-1})\big)R^{sh}(t_{i-1}) + k^{sh}(t_i - t_{i-1})\, b(R(t_{i-1})) + \xi_i \\ Y(t_i) &= R^{sh}(t_i)\, h_0 + \psi(t_i) + \zeta_i \end{aligned} \tag{4.7}$$

We can then compute

$$\widehat{R}^{sh}(t_i) := \mathbb{E}\big[R^{sh}(t_i)\big|\mathcal{F}^Y_{t_i}\big] \tag{4.8}$$

by the Kalman filter:

$$\begin{aligned} &\widehat{R}^{sh}(t_0) := \mathbb{E}\big[R^{sh}(t_0)\big], \quad P_0 := \mathrm{Var}\big[R^{sh}(t_0)\big]; \\ &\text{given } \widehat{R}^{sh}(t_{i-1}) \text{ and } P_{i-1}, \\ &P_i^- := \big(1 - \overline{k}^{sh}(t_i - t_{i-1})\big)^2 P_{i-1} + \sigma_0^2(t_i - t_{i-1})\widehat{R}^{sh}(t_{i-1}) \\ &G_i := P_i^- h_0^T\big(P_i^- h_0 h_0^T + Q\big)^{-1} \\ &\widehat{R}^{sh}(t_i)^- := \big(1 - \overline{k}^{sh}(t_i - t_{i-1})\big)\widehat{R}^{sh}(t_{i-1}) + k^{sh}(t_i - t_{i-1})\, b(R(t_{i-1})) \\ &\widehat{R}^{sh}(t_i) := \widehat{R}^{sh}(t_i)^- + G_i\big[Y(t_i) - \widehat{R}^{sh}(t_i)^- h_0 - \psi(t_i)\big] \\ &P_i := \big(1 - G_i h_0\big)P_i^-. \end{aligned} \tag{4.9}$$

Remark 4.1 *It follows from* (4.6) *that* ξ_i *is independent of* $R^{sh}(t_{i-1})$ *conditionally on* $\mathcal{F}^Y_{t_{i-1}}$. *Usually in the Kalman filter the random variables* $\{\xi_i\}$ *are supposed to be independent, so that* ξ_i *is independent of* $R^{sh}(t_{i-1})$, *but the proof of the Kalman filter carries over to the case when* ξ_i *is independent of* $R^{sh}(t_{i-1})$ *only conditionally on* $\mathcal{F}^Y_{t_{i-1}}$.

4.2 Parameter calibration

From $\widehat{R}^{sh}(t_i)^-$, defined as in (4.10), we can estimate the value of $Y(t_i)$ by

$$\widehat{Y}(t_i) := \widehat{R}^{sh}(t_i)^-\, h_0 + \psi(t_i), \tag{4.10}$$

and hence obtain the observation error

$$\widehat{E}(t_i) := Y(t_i) - \widehat{Y}(t_i). \tag{4.11}$$

Conditionally on $\mathcal{F}^Y_{t_{i-1}}$, the mean of $\widehat{E}(t_i)$ is zero and its variance is given by

$$F(t_i) := P_i^- h_0 h_0^T + Q, \qquad \text{for } i \geq 2, \tag{4.12}$$

$$F(t_1) := \big\{\big(1 - \overline{k}^{sh} t_1\big)^2 \mathrm{Var}[R^{sh}(0)] + \sigma_0^2 t_1 \mathbb{E}[R^{sh}(0)]\big\} h_0 h_0^T + Q$$

Approximating the law of $\widehat{E}(t_i)$, conditioned on $\mathcal{F}^Y_{t_{i-1}}$, by a Gaussian law, for $i \geq 2$ independent of $\{\widehat{E}(t_h)\}_{h\leq i-1}$, the log-likelihood for the parameters of our model, at time t_i, is

$$L(t_i) := -\frac{iJ}{2}\log(2\pi) - \frac{1}{2}\sum_{h=1}^{i}\log(|F(t_h)|) - \frac{1}{2}\sum_{h=1}^{i}\left(\widehat{E}(t_h)^T F(t_h)^{-1}\widehat{E}(t_h)\right). \tag{4.13}$$

Among the parameters of our model, we fix

$$\pi^* = \ln(1.02), \quad v = \sigma_\Pi,$$

where σ_Π is the historical standard deviation of the monthly increments of the inflation,

$$\underline{r} = 0.05\%, \;\; \overline{r} = 4.5\%, \;\; m = 1, \;\; \delta = 0.25\%,$$

and the probabilities q as

$$q(\pi, r, \delta) = \left[\left(\frac{1}{0.3\sigma_\Pi}\left(\pi - (\pi^* + 0.2\sigma_\Pi)\right)\right)_+ \wedge 1\right]\left[\left(\frac{1}{3\delta}\left((\overline{r} - \delta) - r\right)\right)_+ \wedge 1\right],$$

$$q(\pi, r, -\delta) = \left[\left(\frac{1}{0.3\sigma_\Pi}\left((\pi^* - 0.2\sigma_\Pi) - \pi\right)\right)_+ \wedge 1\right]\left[\left(\frac{1}{3\delta}\left(r - (\underline{r} + \delta)\right)\right)_+ \wedge 1\right],$$

$$q(\pi, r, 0) = 1 - q(\pi, r, \delta) - q(\pi, r, -\delta),$$

and we maximize $L(t_i)$ with respect to the other parameters

$$k^{s+h}, \overline{k}^{sh}, \, b_0, \, b_1, \, \sigma_0, \, \lambda, \, \beta_0, \beta_1, \, k^\Pi.$$

Note that in this way we estimate both the parameters under the risk neutral measure and the risk premium $\vartheta = k^{sh} - \overline{k}^{sh}$.

5 Numerical results

We validate the model on market data from German bonds provided by the Bloomberg platform. The dataset covers ZCB and coupon-bearing bond prices for the period of time March 30th 2007 to December 31st 2015 (106 months), with maturities 6 months, 1, 3, 5, 7, 10 years. Since the ZCB prices are not available daily and are not available for all maturities, in order to obtain the ZCB yield curve, as usual we consider daily market prices of zero and coupon-bearing bonds and we apply the bootstrapping technique. We also apply an interpolation to determine the ZCB yields for an arbitrary maturity T.

For the inflation index Π we use the Harmonized Index of Consumer Prices (HICP). The HICP index and the CB rate are obtained by Eurostat. We recall that the variance parameter of the inflation rate is fixed as the historical standard deviation $v = \sigma_\Pi = 0.2770$. Since the HICP index is available monthly, in order to have an organic set of comparable data, for each month in the sample period, we consider the day where the HICP index is observed and, for each maturity, we extract the corresponding ZCB yield.

Figure 1 shows the filter estimated value of the interest rate R^{sh}, instead Figures 2, 3, 4, 5 and 6 show the comparison between the observed yield rate Y (blue line) and the model implied yield rate $\widehat{Y}$ defined by (4.10) (red line) for the 6-month, 1-year, 3-year, 5-year and 10-year maturities.

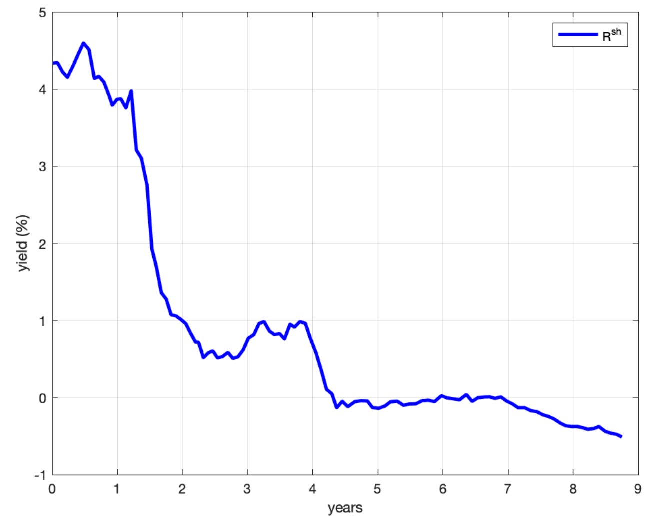

Figure 1: The filter estimated value of the short-term interest rate R^{sh}.

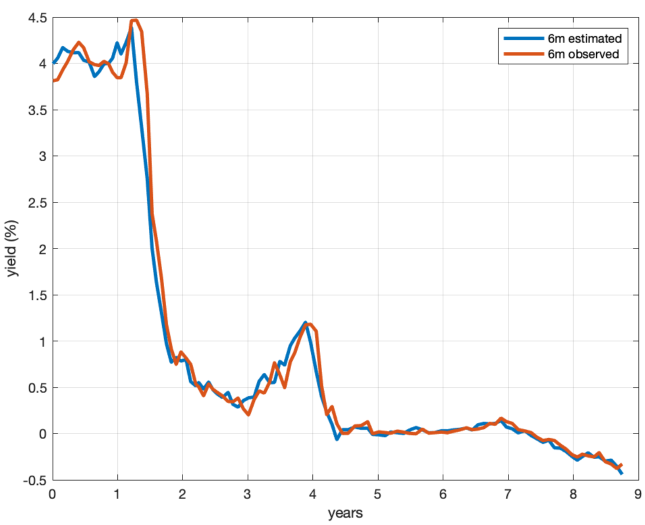

Figure 2: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 6-month maturity.

We use our model, with the parameters progressively calibrated as described in Section 4 to make an out-of-sample forecast. Figure 7 shows the predicted values for the 20-year maturity ZCB, computed using the short-term interest rate and the parameter values estimated by the maturities up to 10 years.

Table 1 reports, for each considered maturity, the root mean-square error from March 2007 (t_0) to October 2015 (t_M):

$$RMSE_j := \sqrt{\sum_{i=1}^{M} \frac{[Y_j(t_i) - \widehat{Y}_j(t_i)]^2}{M}}.$$

From Figures 2, 3, 4, 5,6, 7 and the error table, the $\widehat{Y}$ curve of the model implied ZCB yields appears to follow quite

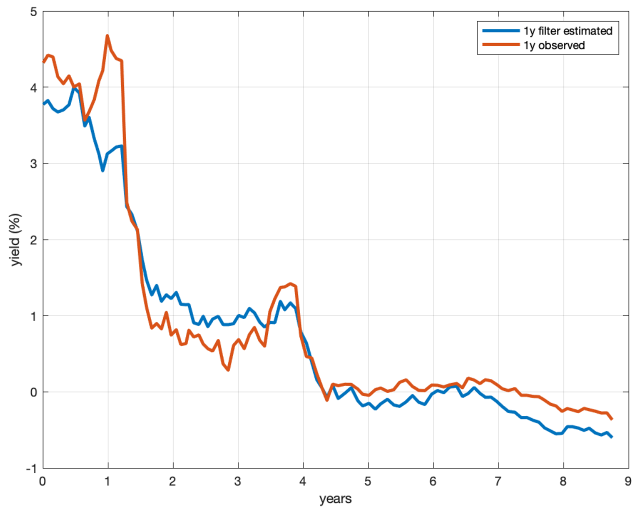

Figure 3: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 1-year maturity.

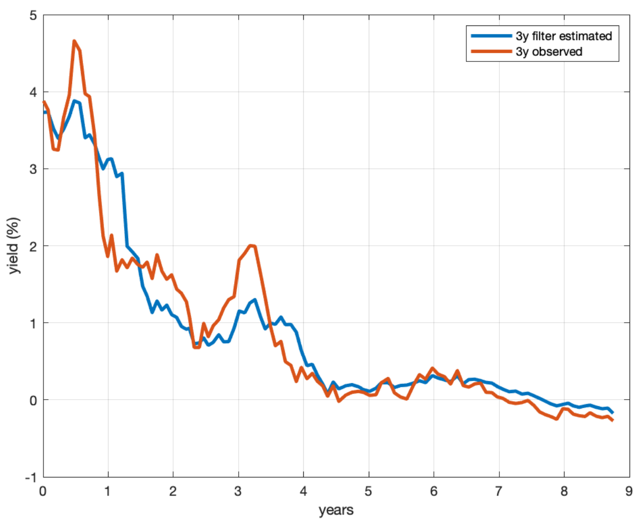

Figure 4: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 3-year maturity.

well the Y curve of the market ZCB yields. Even more significantly, Figure 7 shows that the model has a good predictive power for other bonds not included in the sample used for the calibration. The fit is even more meaningful as the ZCB yields have undergone major changes in the period from March 2007 to December 2015. All these results confirm that our model describes well the market evolution.

Our model is not a "black box" one, but attempts to capture the interactions among inflation, the CB interest rate and the short-term interest rate employing relatively few parameters. Table 2 reports the values of the parameters calibrated using the information for the first 100 months. In particular the b_1 value confirms that the interaction between the short-term interest rate and the CB interest rate cannot be neglected.

We also conduct an *ex-post* analysis of residuals. Precisely, for the dynamics of the inflation rate, we calculate the resid-

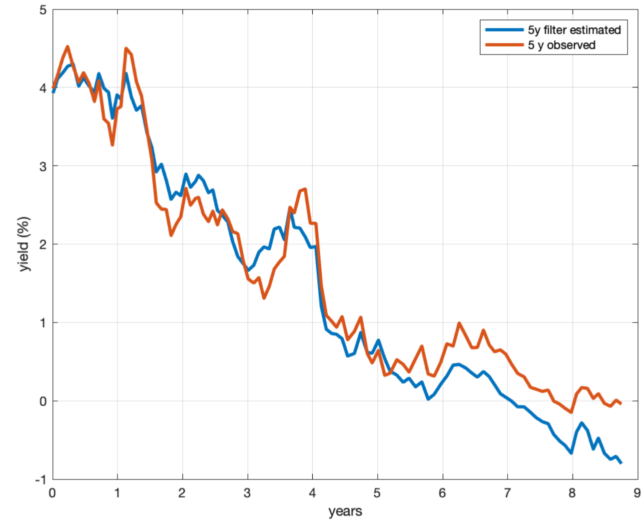

Figure 5: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 5-year maturity.

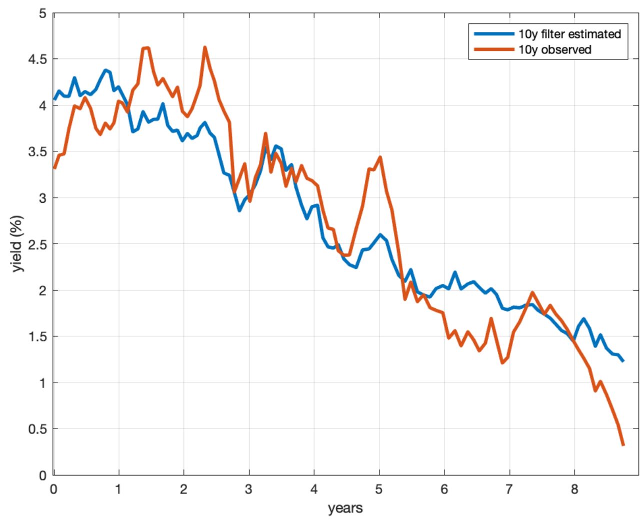

Figure 6: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 10-year maturity.

uals $\eta_{i+1} = \Pi(t_{i+1}) - \beta(\Pi(t_i), R(t_{i+1}^-))$, where $\Pi(t_i)$ and $R(t_i)$ are the observed values of the inflation index and the CB rate and β is computed using the parameter values of Table 2, and we apply the *Jarque-Bera* normality test: The normality null hypothesis cannot be rejected, at a confidence level of 95%. The value of the statistics is 2.5065, with a critical value of 5.4749 and a p-value of 0.2028. Figure 8 shows a comparison between the empirical distribution function of the standardized residuals and the standard normal distribution function. Here, the L^∞-norm of the difference is 0.0763.

Similarly, we analyze the residuals $\zeta_i = Y(t_i) - \widehat{R}^{sh}(t_i)h_0 - \psi(t_i)$, computed using the parameter values of Table 2 and the value of $\widehat{R}^{sh}(t_i)$ provided by the filter. Under the *Jarque-Bera* test for both the 5-year and 10-year bond yield, the null

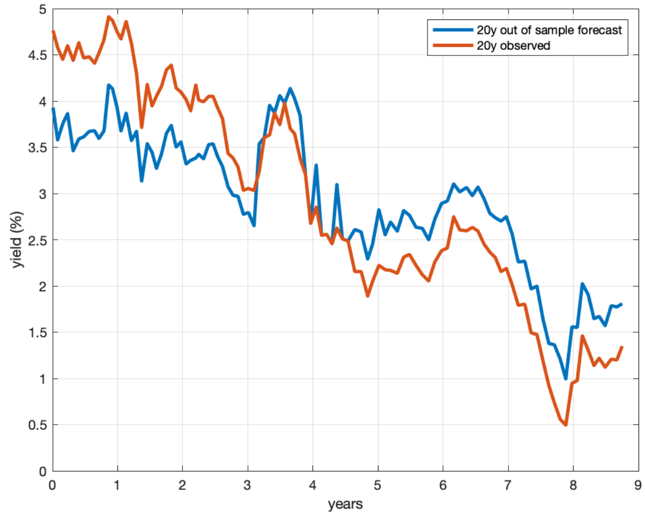

Figure 7: Comparison between the market yield rate (blue line) and the model implied yield rate (red line) for the 20-year maturity.

Maturity	**RMSE**
6-months	0.09673
1-years	0.14272
3-years	0.18721
5-years	0.21734
10-years	0.39351
20-years	0.51213

Table 1: *The root mean-square error for each considered maturity. The last error is for the out-of-sample forecast of the 20-year yield*

Parameter	**Estimated Value**
$\overline{k}^{sh}$	5.5035
k^{sh}	0.6821
b_0	4.3131
b_1	0.8715
σ_0	2.2352
λ	0.14971
k^{Π}	0.01262
β_0	0.86262
β_1	0.02143

Table 2: *The estimated parameters.*

hypothesis of normality cannot be rejected, at a confidence level of 95%. The value of the statistics is 0.5541 for the 5-year bond and 0.7842 for the 10-year bond, with a p-value of 0.5. Figures 9 10 show the $q-q$ plot of the empirical distribution of residuals for both the maturities. The L^{∞}-norm of the difference between the empirical distribution function of the

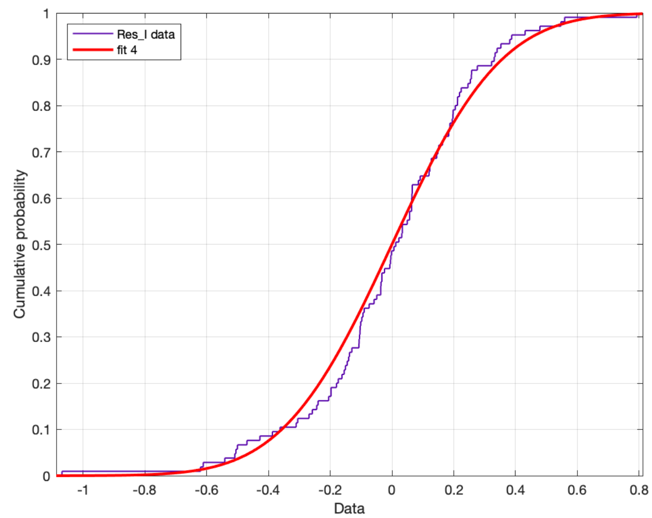

Figure 8: The standard normal distribution (red line) and the empirical distribution (blue line) of the standardized residuals for inflation.

standardized residuals and the standard normal distribution function for both the maturities, is 0.0478 for the 5-year bond and is 0.0653 for the 10-year bond.

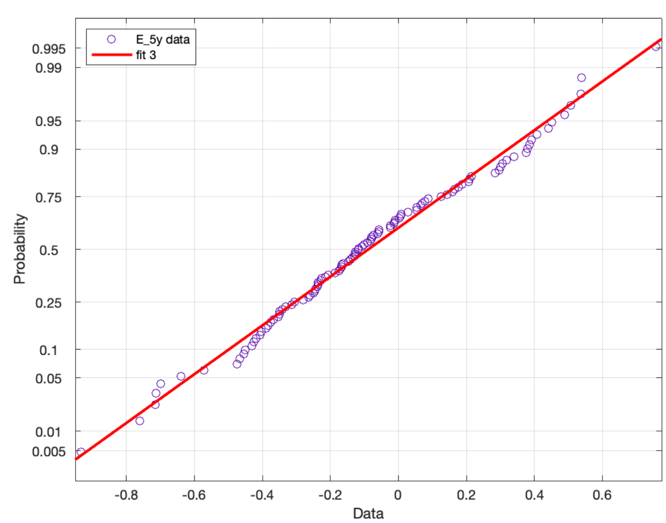

Figure 9: Q-Q plot of the empirical distribution of the standardized residuals for the 5-year bond yield.

6 Conclusions

We have considered the model that we proposed in a previous paper ([ACDP20]) to describe the joint evolution of inflation, the Central Bank interest rate and the short-term interest rate. Our model involves only factors with a clear economic

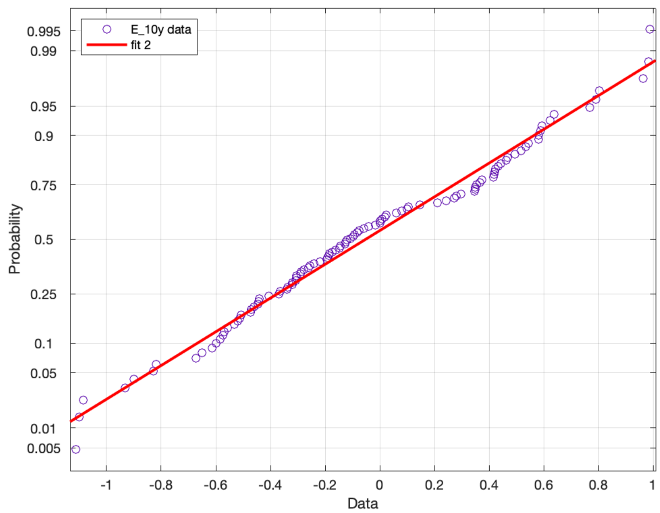

Figure 10: Q-Q plot of the empirical distribution of the standardized residuals for the 10-year bond yield.

interpretation and employs many fewer parameters than the other ones known in the literature. We have shown that, in the case when the diffusion coefficient does not depend on the CB interest rate, our model yields a semiclosed valuation formula for contingent derivatives, in particular for ZCBs.

By using ZCB yields as observations, we have implemented the Kalman filter and we have obtained a dynamical estimate of the short-term interest rate $R^{sh}(t)$. By employing again our valuation formula, this estimate provides a one-step-ahead prediction of the ZCB yields. By using this prediction, at each time step we have made a quasi-maximum likelihood estimation of the model parameters under the risk neutral measure and of the coefficient of the risk premium. Considering the values of the parameters calibrated using the information from the first 100 months, we see that the b_1 value confirms that the interaction between the short-term interest rate and the CB interest rate cannot be neglected.

We have compared the market values of German ZCB yields for several maturities with the corresponding values predicted by our model, from 2007 to 2015. The model implied values follow the market values quite closely, even though the ZCB yields have undergone major changes during the considered period. Even more significantly, our model and filtering/calibration procedure provide quite a good prediction for the 20-year ZCB yield, which is not included in the sample used to estimate the values of the short-term interest rate and of the parameters.

Our estimate of the short-term interest rate $R^{sh}(t)$ can be used to evaluate more sophisticated fixed-income derivative instruments (for example credit risk derivatives): We intend to pursue this in the future.

Appendix

In (4.10), in order to compute $\widehat{R}^{sh}(t_{i+1})$, we have to compute $\psi_j(t_{i+1})$, $j = 1, ..., J$, by using (3.10), with $\alpha_0 = 0$, and by solving numerically the system (3.10)-(3.11)-(3.12) with $p = 0$ and $\Phi_0 = 1$. In this appendix we describe the numerical procedure that we set up to do this.

In the sequel we omit the subscript j and write τ for t_{i+1}. Taking into account that $\{t_i\}_{i=1,...,M} \subseteq \{t_i\}_{n=1,...,N}$, if $t_l \le \tau < t_{l+1}$, $t_L \le \tau + T < t_{L+1}$, $l \le L \le M$, this amounts to solving, for each π:

- for $i = L$,

$$\begin{aligned}\frac{\partial \varphi_0^L}{\partial s}(s,\pi,r) &+ \left(k^{sh} b(r)\alpha(t_i+s)\right)\varphi_0^L(s,\pi,r)\\ &+ \lambda \sum_{h=-m}^{m} \left[\varphi_0^L(s,\pi,r+h\delta) - \varphi_0^L(s,\pi,r)\right] q(\pi,r,h\delta)\\ &= 0, s\in[0,\tau+T-t_L),\end{aligned}$$

with terminal condition

$$\varphi_0^L(\tau+T-t_L,\pi,r) = 1;$$

- for $i = l+1, ..., L-1$,

$$\begin{aligned}\frac{\partial \varphi_0^i}{\partial s}(s,r) &+ \left(k^{sh} b(r)\alpha(t_i+s)\right)\varphi_0^i(s,r)\\ &+ \lambda \sum_{h=-m}^{m} \left[\varphi_0^i(s,r+h\delta) - \varphi_0^i(s,r)\right] q(\pi,r,h\delta)\\ &= 0, s\in[0,t_1)\end{aligned}$$

with terminal condition

$$\varphi_0^i(t_1,\pi,r) = \int_{\mathbb{R}} \varphi_0^{i+1}(0,\beta(\pi,r)+u,r)\,\mathcal{N}_{0,v^2}(u)du,$$

where $\mathcal{N}_{0,v^2}(u)$ denotes the Gaussian density of zero mean and variance v^2;

- for $i = l$,

$$\begin{aligned}\frac{\partial \varphi_0^l}{\partial s}(s,\pi,r) &+ \left(k^{sh} b(r)\alpha(t_i+s)\right)\varphi_0^l(s,\pi,r)\\ &+ \lambda \sum_{h=-m}^{m} \left[\varphi_0^l(s,\pi,r+h\delta) - \varphi_0^l(s,\pi,r)\right] q(\pi,r,h\delta)\\ &= 0, s\in[0,t_{l+1}-\tau)\end{aligned}$$

with terminal condition

$$\varphi_0^l(t_{l+1}-\tau,\pi,r) = \int_{\mathbb{R}} \varphi_0^{l+1}(0,\beta(\pi,r)+u,r)\,\mathcal{N}_{0,v^2}(u)du.$$

Then

$$\psi(\tau) = -\frac{1}{T}\log\left(\varphi_0^l(0,\Pi(\tau),R(\tau))\right).$$

From now on we consider only the equation for $i = l+1, ..., L-1$: the computations for $i = l$ and $i = L$ are completely analogous. In addition we omit the subscript 0.

We consider values of r in the discrete, increasing set $\{r_k\}_{k=1,...,N_r}$, where, $r_{k+1} - r_k = \Delta r := \frac{\delta}{n_r}$, $n_r \in \mathbb{N}$, and $N_r := \left\lfloor \frac{\overline{r}-\underline{r}}{\Delta r} \right\rfloor$. Setting $\varphi_k^i(t,\pi) := \varphi_0^i(t,\pi,k\Delta r)$, we obtain, for each π, the following system of backward ordinary differential equations:

$$\begin{aligned}\frac{d\varphi_k^i}{ds}(s,\pi) &+ \left(k^{sh} b(r_k)\alpha(t_i+s)\right)\varphi_k^i(s,\pi)\\ &+ \lambda \sum_{h=-m}^{m} \left[\varphi_{k+hn_r}^i(s,\pi) - \varphi_k^i(s,\pi)\right] q(\pi,r_k,hn_r\Delta r)\\ &= 0, s\in[0,t_1).\end{aligned}$$

By inverting the time direction, and defining $\phi^i(s,\pi)$ as the vector of components $\varphi^i_k(t_1-s,\pi)$, the above system of ordinary differential equations can be rewritten as

$$\frac{d\phi^i}{ds} = A(s,\pi)\phi^i(s,\pi), \qquad s\in[0,t_1), \tag{A.1}$$

where

$$\begin{aligned} A_{kk}(s,\pi) &:= -k^{sh}b(r_k)\alpha(t_i+s)+\lambda\big(1-q(\pi,r_k,0)\big),\\ A_{kj}(s,\pi) &:= -\lambda q(\pi,r_k,hn_r\Delta r), \quad \text{for } |j-k|=hn_r, \quad h=1,...,m,\\ A_{kj}(s,\pi) &:= 0, \quad \text{for } |j-k|\neq hn_r, \quad \forall h=1,...,m. \end{aligned} \tag{A.2}$$

The initial datum is

$$\phi^i_k(0,\pi) = \int_{\mathbb{R}} \phi^{i+1}_k(t_1,\beta(\pi,r_k)+u)\,\mathcal{N}_{0,v^2}(u)\,du. \tag{A.3}$$

It would be natural to approximate the integral in $(A.3)$ by a sum of the form

$$\sum_{d=1}^{D}\mathcal{N}_{0,v^2}(u_d)\phi^{i+1}_k(t_1,\beta(\pi,r_k)+u_d)\Delta u, \qquad |u_d|\le 3v, \tag{A.4}$$

where Δu is the discretization step for the variable u. To this end, for each π and k, we would need to compute the value of ϕ^{i+1}_k on the D points $\beta(\pi,r_k)+u_1,...,\beta(\pi,r_k)+u_D$, which in turn would require to compute the value of ϕ^{i+2}_k on the D^2 points $\beta\big(\beta(\pi,r_k)+u_1,r_k\big)+u_1,\ \dots\ ,\beta\big(\beta(\pi,r_k)+u_1,r_k\big)+u_D,\ \dots\ ,\beta\big(\beta(\pi,r_k)+u_D,r_k\big)+u_1,\ \dots\ ,\beta\big(\beta(\pi,r_k)+u_D,r_k\big)+u_D$, and so on, so that, for each pair (π,k), we would need to compute ϕ^L_k on a grid of D^{L-i} points, and a different one for each (π,k).

In order to simplify the computation, we define, for each π, independently of k, a sequence of $L-i$ grids $\{\pi^{i+1}_1,...,\pi^{i+1}_{H_{i+1}}\}$, ... , $\{\pi^L_1,...,\pi^L_{H_L}\}$ constructed in the following way:

the grid step is $\Delta\pi$, constant and the same for all the grids;

$$\begin{aligned} \pi^{i+1}_1 = ... = \pi^L_1 &= k^\pi\pi^* + \underline{\beta} - 3v,\\ \pi^{i+1}_{H_{i+1}} &= \pi + (k^\pi\pi^* + \overline{\beta} + 3v),\\ \pi^{i+2}_{H_{i+2}} &= \pi + 2(k^\pi\pi^* + \overline{\beta} + 3v),\\ &\dots\dots\dots\\ \pi^L_{H_L} &= \pi + (L-i)(k^\pi\pi^* + \overline{\beta} + 3v), \end{aligned}$$

where we consider, for numerical purposes, only values $\pi \ge \underline{\pi}$, $\underline{\pi}\le 0$, and $\underline{\beta} := (\beta_0 - k^\Pi)\underline{\pi} + \min(\beta_1\underline{r},\beta_1\overline{r})$, $\overline{\beta} := (\beta_0 - k^\Pi - 1)\underline{\pi} + \max(\beta_1\underline{r},\beta_1\overline{r})$. β_0, β_1, k^Π and π^* are the parameters in $(refeq:beta)$, and $(\underline{r},\overline{r})$ is the interval in which the variable r takes values ($\underline{r}\le 0$, $\overline{r}>0$). Taking into account that $u_d\in[-3v,3v]$, for all $r\in(\underline{r},\overline{r})$,

$$\pi^{i+1}_1 = ... = \pi^L_1 = k^\pi\pi^* + \underline{\beta} - 3v \le \beta(\pi,r)+u_d \le \pi + k^\pi\pi^* + \overline{\beta} + 3v = \pi^{i+1}_{H_{i+1}} \le \ ...\ \le \pi^L_{H_L},$$

so that for each r_k there is an index d^* such that

$$\beta(\pi,r_k)+u_d \in [\pi^{i+1}_{d^*},\pi^{i+1}_{d^*+1}].$$

Then we can replace $\phi^{i+1}_k(t_1,\beta(\pi,r_k)+u_d)$ by a linear interpolation between $\phi^{i+1}_k(t_1,\pi^{i+1}_{d^*})$ and $\phi^{i+1}_k(t_1,\pi^{i+1}_{d^*+1})$. This procedure is repeated up to L. In this way, at step j we need to compute ϕ^j_k only on the level j grid points. As the number of the grid points increases linearly in j, the total number of points in all grids is of the order of $(L-i)(L-i+1)/2$.

Finally we solve the system -$(A.1)(A.2)$ by a standard routine.

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