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Output transformations and separation results for feedback linearisable delay systems

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ABSTRACT

The class of strict-feedback systems enjoys special properties that make it similar to linear systems. This paper proves that such a class is equivalent, under a change of coordinates, to the wider class of feedback linearisable systems with multiplicative input, when the multiplicative terms are functions of the measured variables only. We apply this result to the control problem of feedback linearisable nonlinear MIMO systems with input and/or output delays. In this way, we provide sufficient conditions under which a separation result holds for output feedback control and moreover a predictor-based controller exists. When these conditions are satisfied, we obtain that the existence of stabilising controllers for arbitrarily large delays in the input and/or the output can be proved for a wider class of systems than previously known.

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1. Introduction

The control problem of nonlinear systems affected by input and/or output delays has attracted considerable research efforts in recent years and many results are now available in the literature (for example Datko, 1988; Fridman, 2014; Henson & Seborg, 1994; Hu & Rangaiah, 1999; Jankovic, 2001; Kravaris & Wright, 1989; Mazenc & Bliman, 2006; Ortega & Lozano, 1988; Zhong, 2006). In particular, control by state feedback in the presence of input delays by means of predictors has been thoroughly investigated (see for example Bekiaris-Liberis & Krstic, 2012; Bekiaris-Liberis & Kritic, 2013a, 2013b; Karafyllis, 2011; Karafyllis & Krstic, 2012, 2013; Karafyllis, Krstic, Ahmed-Ali, & Lamnabhi-Lagarrigue, 2014; Krstic, 2009, 2010b; Lei & Khalil, 2016a, and the survey in Krstic, 2010a), for several classes of systems, delay functions, disturbances, etc. The stabilisation problem in the presence of arbitrarily large input delays has been proved feasible for certain classes of systems (Cacace, Conte, Germani, & Pepe, 2016b; Karafyllis & Krstic, 2012; Mazenc & Malisoff, 2016; Mazenc, Mondie, & Francisco, 2004), either by means of infinite dimensional predictors or through a cascade of finite dimensional ones. The dual problem of state observation from delayed measurements has also been solved under similar hypotheses for arbitrarily large delays (Ahmed-Ali, Cherrier, & Lamnabhi-Lagarrigue, 2012; Cacace, Germani, & Manes, 2014).

When a system is affected by both input and output delays, it is necessary to pursue some separation result in order to ensure that the resulting output feedback control still enjoys the same global properties as the observer and predictor on which it is based. This issue has not received much attention so far, even if specific results exist. The recent paper (Lei & Khalil, 2016b) introduces the use of a high-gain-predictor for realising stabilising output feedback for systems affected by both input and output delays which is able to recover the performance of state feedback, but makes the high-gain parameter dependent from the delay bound. In the context of strict-feedback delay systems, that is feedback linearisable systems with purely additive input, the results in Cacace, Germani, and Manes (2015) and Cacace et al. (2016b) prove that the separation principle holds (see also Karafyllis, 2011; Karafyllis & Krstic, 2013; Oguchi, Watanabe, & Nakamizo, 2002 for more results on this class of systems). However, less restrictive sufficient separability conditions exist for delay-free systems (Cacace et al., 2015).

In this paper we consider the following two problems:

- given a nonlinear system without state delays that admits a globally stabilising state feedback controller, design a state observer based on delayed measurements such that the resulting output feedback controller is globally stabilising;

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- given a nonlinear system admitting a that admits a state feedback controller in the delay-free case, design a state predictor such that the resulting predictor-based state feedback controller is stabilising in the presence of input delay.

The solutions to the above two problems can be combined, resulting in a globally stabilising observer+predictor output feedback controller when both input and output delays are present. We remark that the above problem have already been solved in the case of strict-feedback delay systems, see for example (Cacace et al., 2015, 2016b). In this paper we extend the results presented in these works to the more general class of feedback linearisable delay systems, where the input term is multiplicative. Section 2 describes the problems arising in this case. Section 3 contains the original contribution of the paper. We provide sufficient conditions for both the single-input single-output (SISO) and multi-input multi-output (MIMO) cases that extend the results of strict-feedback systems to the more generic feedback linearisable ones. In these conditions, the two problems above can be solved for delayed systems whenever they are solvable for their un-delayed counterpart. Section 4 provides two examples of application to output feedback control and predictor-based control. Conclusions follow in Section 5.

Notation: I_n indicates the identity matrix in $\mathbb{R}^n$. (A_b, B_b, C_b) denotes a Brunowski matrices triple of size n. Given a multi-index $\bar{r} = \{r_1, r_2, \ldots, r_q\}$, $(A_{\bar{r}}, B_{\bar{r}}, C_{\bar{r}})$ denotes a triple of diagonal block matrices composed by the respective Brunowski blocks of sizes r_i, $i = 1, 2, \ldots, q$. For a square matrix A, $\rho(A)$ denotes the largest real part of its eigenvalues. If $\rho(A) < 0$ the matrix is Hurwitz.

Given a smooth vector field $\phi : \mathbb{R}^n \to \tilde{}\mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}^s$, the Lie derivative $L_\phi g : \mathbb{R}^n \to \mathbb{R}^s$ is defined as

$$L_\phi g = \frac{dg}{dx}\phi. \tag{1}$$

$L^i_\phi g = L_\phi(L^{i-1}_\phi g)$, $i \geq 1$, denotes the ith Lie derivative, with $L^0_\phi g = g$. Throughput the paper, the time argument will be omitted when not explicitly needed.

2. The separation principle for nonlinear systems

In this section, we recall the separation principle and some conditions under which it is satisfied first for delay-free nonlinear systems and then for the two cases of feedback from delayed outputs and state feedback with delayed input.

2.1 Output feedback control

A general approach to the output feedback control problem is to use a state feedback controller that achieves stabilisation together with a state observer that provides an estimate converging to the true state value. When it is possible to separately design these two components, while retaining the stabilisation properties of the controller for known states, the system is said to satisfy a *separation principle*. It is well known that the separation principle holds for linear systems, but, in general, it does not for nonlinear systems (Khalil, 2015).

Feedback linearisable nonlinear systems admit sufficient conditions that ensure the separation principle. To illustrate the problem and these conditions, consider a SISO nonlinear affine system:

$$\dot{x} = f(x) + g(x)u \tag{2}$$
$$y = h(x), \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $u(t) \in \mathbb{R}$ is a scalar input and $y(t) \in \mathbb{R}$ is a scalar output.

Definition 2.1: Given a function $\lambda : \mathbb{R}^n \to \mathbb{R}$, if for any $x \in \mathbb{R}^n$, $L_g L^i_f \lambda(x) = 0$, for all $i = 0, 1, \ldots, r-2$ and $L_g L^{r-1}_f \lambda(x) \neq 0$, then system (2) is said to have relative degree r with respect to $\lambda(x)$. If system (2)–(3) has relative degree n with respect to $h(x)$ is said to have full relative degree.

Lemma 2.1 (Isidori, 2005)**:** *If: (a) system (2)–(3) has full relative degree and (b) the observability map*

$$z = \Theta(x) = [h(x),\ L_f h(x),\ \ldots,\ L^{n-1}_f h(x)]^T \tag{4}$$

is a global(local) diffeomorphism, then system (2)–(3) can be written as

$$\dot{z} = A_b z + B_b\left(a(x) + b(x)u\right), \quad x = \Theta^{-1}(z) \tag{5}$$
$$y = C_b z, \tag{6}$$

with

$$a(x) = L^n_f h(x), \tag{7}$$
$$b(x) = L_g L^{n-1}_f h(x), \tag{8}$$

and it admits the global(local) stabilising state feedback

$$u = -\frac{K\Theta(x) + a(x)}{b(x)}, \tag{9}$$

where K is such that $A_b - B_b K$ is Hurwitz.

When the output y is available, but the state x is not, we need to resort to an observer. The high-gain observer for (5) has the form,

$$\dot{\hat{z}} = A_b\hat{z} + B_b(a_z(\hat{z}) + b_z(\hat{z})u) + L\left(y - C_b\hat{z}\right) \quad (10)$$
$$a_z(z) = a(\Theta^{-1}(z)) \quad (11)$$
$$b_z(z) = b(\Theta^{-1}(z)). \quad (12)$$

It is possible to find L that makes (10) an exponential observer in open-loop.

Lemma 2.2: *Under the same hypotheses of Lemma 2.1, if a_z and b_z are uniformly Lipschitz with respect to z and u is bounded, for any chosen exponential decay rate there exists L so that (10) is a global(local) exponential observer of (5) with the prescribed exponential decay rate.*

Proof: The result of Lemma 2.2 is well known, see for example, Khalil (2015) or Ciccarella, Dalla Mora, and Germani (1993) where the observer is written in the original coordinates. ∎

Remark 2.1: The key point is that, even in the favourable case when both the control and the observer are global, the resulting output feedback control is in general not global. In fact, replacing x with $\hat{x} = \Theta^{-1}(\hat{z})$ in (9) we obtain a control law that can be implemented. However, the resulting dynamics of the observer estimation error $\varepsilon = z - \hat{z}$ becomes

$$\dot{\varepsilon} = (A_b - LC_b)\varepsilon + B_b\left(a_z(z) - \frac{b_z(z)}{b_z(\hat{z})}a_z(\hat{z}) + \left(1 - \frac{b_z(z)}{b_z(\hat{z})}\right)K\hat{z}\right), \quad (13)$$

that clearly shows a critical dependence of ϵ from the control gain K. In other words, the closed-loop estimation error depends on the control gain, the separation principle does not hold and the resulting controller is not globally stabilising.

As mentioned above, in the case with no delays it is possible to give simple sufficient conditions for a globally stabilising output feedback controller (Cacace et al., 2015), as stated in the next result.

Lemma 2.3: *Under the same hypotheses of Lemma 2.2, if there exists a function $\mu : \mathbb{R} \to \mathbb{R}$ such that $b(x) = L_gL_f^{n-1}h(x) = \mu(y)$ for all $x \in \mathbb{R}^n$, then the output feedback controller obtained by combining the observer (10)–(12) with the control law (9), computed in the observer estimate $\hat{x}$, is a globally stabilising controller if K is the same as in the state feedback case and L is designed to achieve global convergence to zero of the observation error.*

Proof: The thesis holds when the observer estimation error ϵ does not depend on the control gain K. From (4) it follows that $z_1 = y$ is a measured signal. Moreover, from $b(x) = \mu(y)$ it results that $b_z(z) = b(\Theta^{-1}(z)) = \mu(y) = \mu(z_1)$. It is therefore possible to set $b_z(\hat{z}) = b_z(z)$ and obtain from (13) that

$$\dot{\varepsilon} = (A_b - LC_b)\varepsilon + B_b\left(a_z(z) - a_z(\hat{z})\right). \quad (14)$$

∎

Remark 2.2: In essence, Lemma 2.3 states that, in the case of feedback linearisable systems, the separation principle is satisfied when $b(x) = L_gL_f^{n-1}h(x)$ is a function of the measured output only. Similar conditions hold for the MIMO case.

2.2 Delayed output: output feedback control

Consider nonlinear affine system with delayed output of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad (15)$$
$$y(t) = h(x(t - \delta(t))), \quad (16)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $h : \mathbb{R}^n \to \mathbb{R}^q$ and $\delta(t) \in [0, \Delta]$ is a known function. In this section, we consider the SISO case, i.e. $p = q = 1$.

Since the measurements are affected by time delay, the design of an output feedback stabilisation requires a state observer $\hat{x}(t) = \Gamma(y(t - \delta(t)))$, which estimates the current state $x(t)$ by using delayed measurements. $\hat{x}(t)$ is then used to generate a stabilising state feedback. As highlighted in Cacace et al. (2015), Lemma 2.3 cannot be used for systems with delayed output, since $z_1(t)$ is no more available. Theorem 2 in Cacace et al. (2015) proves that separation is satisfied for strict-feedback systems.

Theorem 2.1 (Cacace et al., 2015)**:** *If: (a) system (15)–(16) with $p = q = 1$ has full relative degree, (b) the observability map (4) is a global(local) diffeomorphism, and (c) the system under $z = \Theta(x)$ becomes*

$$\dot{z}(t) = A_bz(t) + B_b(a(z(t)) + bu(t)) \quad (17)$$
$$y(t) = C_bz(t - \delta(t)), \quad (18)$$

with $a(z)$ uniformly Lipschitz in the domain of interest and $L_gL_f^{n-1}h(x) = b$ being a non-null constant, then there exists a state observer $\hat{x}(t) = \Gamma(y(t - \delta(t)))$ such that system (15)–(16) is globally(locally) exponentially stabilised with arbitrary decay rate by the control law

$$u(t) = -\frac{K\Theta(\hat{x}(t)) + a(\Theta(\hat{x}(t)))}{b}, \quad (19)$$

where K is such that $A_b - B_bK$ is Hurwitz.

Remark 2.3: Note that the condition of Theorem 2.1 that $b(x) = L_gL_f^{n-1}h(x)$ is a non-null constant b is more restrictive than in the delay-free case of Lemma 2.3, where $b(x)$ is allowed to be a function of the output.

We finally remark that, when the hypotheses of Theorem 2.1 hold, the system can be globally stabilised for output delay functions with arbitrary bound on the delay.

2.3 Input delay: predictor-based state feedback

Consider a nonlinear affine system with delayed input of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t - \delta(t)), \tag{20}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, and $\delta(t) \in [0\,\Delta]$ is a known function. The state $x(t)$ is assumed to be available at time t. In this section, we consider the single-input case, i.e. $p = 1$.

A predictor-based state feedback control employs a predictor or a cascade of predictors that at time t estimates the future state $x(t + \delta(t))$ using the current one $x(t)$. Throughput the paper, we will indicate the prediction as $\xi(t) = \Gamma(x(t))$. The state prediction is then used to generate a stabilising state feedback.

For the class of nonlinear strict-feedback systems, it is possible to design a state predictor and a globally stabilising state feedback controller to obtain a globally stabilising controller in presence of input delays (Cacace et al., 2016b). Next theorem summarises the main result of that paper.

Theorem 2.2 (***Cacace et al., 2016b***)**:** *If there exists a global(local) diffeomorphism $z = \Theta(x)$ through which system (20) with $p = 1$ can be rewritten as*

$$\dot{z}(t) = A_b z(t) + B_b(a(z(t)) + bu(t - \delta(t))), \tag{21}$$

with $a(z)$ uniformly Lipschitz in the domain of interest and $b \neq 0$, then there exists a predictor $\xi(t) = \Gamma(x(t))$ of $z(t + \delta(t))$, such that system (20) is globally(locally) stabilised for any delay with arbitrary decay rate by the control law

$$u(t) = -\frac{K\xi(t) + a(\xi(t))}{b}, \tag{22}$$

where K is such that $A_b - B_b K$ is Hurwitz.

Remark 2.4: As in the case of delayed-output feedback, the approach is feasible when the change of coordinates $z = \Theta(x)$ makes the input gain a non-null constant b.

Remark 2.5: Theorem 2.2 implies that for the class of feedback linearisable systems, i.e. systems having relative degree n with respect to $\lambda(x) = x_\ell$, $\ell \in \{1, 2, \dots, n\}$, the predictor-based controller is feasible if the map

$$z = \Theta(x) = [\lambda(x),\ L_f\lambda(x), \dots,\ L_f^{n-1}\lambda(x)]^T \tag{23}$$

is a global(local) diffeomorphism and $b = L_g L_f^{n-1}\lambda(x)$ is a non-null constant.

We finally remark that when the hypotheses of Theorem 2.2 hold the system can be stabilised in the presence of known input delay functions with arbitrary delay bound.

2.4 The MIMO cases

The results recalled in the current section can be extended to the MIMO case. To this end we need first to extend the definitions of relative degree given in Section 2.1. Let us consider system (2)–(3) with $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$. Therefore, $g(x) \in \mathbb{R}^{n\times p}$ and $h(x) = [h_1(x), \dots, h_q(x)]^T \in \mathbb{R}^q$. We denote by r_i the *observation relative degree* of the ith output function of system (2)–(3), defined following Definition 2.1 with respect to $h_i(x)$, $i = 1, 2, \dots, q$. The multi-index $\bar{r} = \{r_1, \dots, r_q\}$, that collects all such observation relative degrees, is defined as the *vector observation relative degree* (Dalla Mora, Germani, & Manes, 2000).

Theorem 2.1 can be rewritten in the MIMO case as it follows.

Theorem 2.3: *Assume that system (15)–(16) with $p = q$ has vector observation relative degree $\bar{r}$ such that $r_1 + r_2 + \dots + r_q = n$. If the observability map*

$$\begin{aligned} z &= \Theta(x) \\ &= [h_1(x),\ \dots,\ L_f^{r_1-1}h_1(x),\ \dots,\ h_q(x),\ \dots,\ L_f^{r_q-1}h_q(x)]^T \end{aligned} \tag{24}$$

is a global(local) diffeomorphism and the system can be rewritten through $z = \Theta(x)$ as

$$\dot{z}(t) = A_{\bar{r}} z(t) + B_{\bar{r}}(a(z(t)) + Ru(t)) \tag{25}$$

$$y(t) = C_{\bar{r}}(t - \delta(t)), \tag{26}$$

where $a(z)$ is uniformly Lipschitz in the domain of interest and $R \in \mathbb{R}^{q\times q}$ is a non-singular constant matrix, then there exists a state observer $\hat{x}(t) = \Gamma(y(t - \delta(t)))$ such that system (15)–(16) is globally (locally) exponentially stabilised, with arbitrary decay rate, by the control law

$$u(t) = -R^{-1}\left[K\Theta\left(\hat{x}(t)\right) + a\left(\Theta\left(\hat{x}(t)\right)\right)\right], \tag{27}$$

where K is such that $A_{\bar{r}} - B_{\bar{r}}K$ is Hurwitz.

The result given above for $p = q$ can be easily extended to the case $p \geq q$. For input delays, Theorem 2.2 can be extended to the multi-input case as it follows.

Theorem 2.4: *Assume that there exists a global(local) diffeomorphism $z = \Theta(x)$ through which system (20) is rewritten as*

$$\dot{z}(t) = A_{\bar{r}}z(t) + B_{\bar{r}}(a(z(t)) + Ru(t - \delta(t))), \quad (28)$$

for a multi-index $\bar{r} = \{r_1, r_2, \dots, r_p\}$ such that $r_1 + r_2 + \cdots + r_p = n$, and with $a(z)$ uniformly Lipschitz in the domain of interest and $R \in \mathbb{R}^{p\times p}$ being a non-singular constant matrix. Then, there exists a predictor $\xi(t) = \Gamma(x(t))$ of $z(t + \delta(t))$ such that system (20) is globally (locally) stabilised, with arbitrary decay rate, by the control law

$$u(t) = -R^{-1}(K\xi(t) + a(\xi(t))), \quad (29)$$

where K is such that $A_{\bar{r}} - B_{\bar{r}}K$ is Hurwitz.

Remark 2.6: In analogy with the SISO case, Theorems 2.3 and 2.4 can be applied to strict-feedback MIMO systems. Moreover, in both cases the system can be globally stabilised in presence of input or output delay functions with arbitrary delay bounds.

3. Global stabilisation and separation principle for nonlinear systems with input and/or output delays

Theorems 2.1 and 2.2 provide sufficient conditions for stabilisation in the cases of delayed output feedback and delayed-input state feedback, respectively. For both the cases, the key condition is that $b(x) = L_gL_f^{n-1}h(x)$ (or $L_gL_f^{n-1}\lambda(x)$ for delayed-input state feedback) is a non-null constant. This is a restriction with respect to the delay-free case, in which the same result is obtained under the less restrictive condition that $b(x)$ is a function of the output y only.

In this section we show how this restriction can be overcome by means of output transformations. Note that even if it is possible, under non-restrictive condition, to find a change of coordinates such that a nonlinear system in the form (2) has only an additive input term (Cacace, Conte, Germani, & Palombo, 2016a), it is in general impossible to obtain this while preserving the linearising input structure.

3.1 Delayed output feedback control

Consider a feedback linearisable nonlinear system with delayed output of the form (15)–(16) with $p = q = 1$ (SISO case). We assume that the system satisfies the following hypotheses :

$\mathcal{H}_{1.1}$: it has full relative degree;

$\mathcal{H}_{1.2}$: the observability map $z = \Theta(x)$ defined in (4) is a global diffeomorphism;

$\mathcal{H}_{1.3}$: there exists $\mu : \mathbb{R} \to \mathbb{R}$ such $b(x) = L_gL_f^{n-1}h(x) = \mu(h(x)) \neq 0$ for all $x \in \mathbb{R}^n$.

Remark 3.1: As discussed in Section 2.1, $\mathcal{H}_{1.1}$–$\mathcal{H}_{1.3}$ are required to design a global high-gain observer and globally stabilise the system by applying separation in the delay-free case.

We prove the existence of an output post-processing, consisting of a smooth and invertible function $\phi : \mathbb{R} \to \mathbb{R}$, designed to satisfy the separation principle stated in Theorem 2.1 by replacing the original output y with

$$\bar{y} = \phi(y). \quad (30)$$

The corresponding output function is

$$\bar{h}(x) = \phi(h(x)). \quad (31)$$

Of course, both $y(t) = h(x(t - \delta(t)))$ and $\bar{y}(t) = \bar{h}(x(t - delta(t)))$ are functions of the delayed state, and they are available at time t. Our aim is to find ϕ so that the corresponding map based on the time derivatives of $\phi(h(x))$ does not contain anymore the term $\mu(h(x(t)))$, which, in the delayed case, is not available. If this holds true, the separation principle can be restored.

In the following $\phi^{\langle i\rangle}$ indicates the ith derivative of ϕ with respect to y and, given any time-varying vector v, $v^{(i)}$ indicates the ith time derivative of v, $v^{(0)} = v$. Consequently,

$$L_f\phi^{\langle i\rangle} = \frac{d\phi^{\langle i\rangle}}{dx}f = \frac{d\phi^{(i)}}{dy}\frac{dh}{dx}f = \phi^{\langle i+1\rangle}L_fh. \quad (32)$$

As stated by the following lemma, the Lie derivatives $L_f^i\phi^{\langle 1\rangle}$ can be expressed as a function of the $\phi^{(j)}$ with $j \leq i + 1$ and of the L_f^kh with $k \leq i$.

Lemma 3.1:

$$L_f^i\bar{h} = \sum_{j=1}^{i}\binom{i-1}{j-1}L_f^{i-j}\phi^{\langle 1\rangle}L_f^jh. \quad (33)$$

Proof: For $i = 1$ (32) satisfies (33). For $i > 1$, (33) can be verified inductively by using

$$L_f\left(L_f^\alpha\phi^{\langle 1\rangle}L_f^\beta h\right) = L_f^{\alpha+1}\phi^{\langle 1\rangle}L_f^\beta h + L_f^\alpha\phi^{\langle 1\rangle}L_f^{\beta+1}h \quad (34)$$

and the properties of the binomial coefficients. ■

We define now the output transformation ϕ as

$$\phi(y) = \int \frac{1}{\mu(y)}dy, \quad (35)$$

and the corresponding observability map as

$$w = \overline{\Theta}(x) = [\bar{h}(x), L_f\bar{h}(x), \ldots, L_f^{n-1}\bar{h}(x)]^T. \quad (36)$$

Remark 3.2: Note that with the choice (35) the function $\bar{y} = \phi(y)$ can be inverted, thanks to the fact that $\mu(y) \neq 0$ as assumed in $\mathcal{H}_{1.3}$.

A key property of $\overline{\Theta}$ is formalised by the following lemma.

Lemma 3.2: *If system (15)–(16) with $p = q = 1$ satisfies $\mathcal{H}_{1.1}$–$\mathcal{H}_{1.3}$ and ϕ is chosen as in (35), then $w = \overline{\Theta}(x)$ is a local diffeomorphism.*

Proof: Since by $\mathcal{H}_{1.2}$ $z = \Theta(x)$ is a global diffeomorphism, the thesis is proved if we show that $w = \widetilde{\Theta}(z)$ is a local diffeomorphism, that is, $dw/dz \neq 0$. To this end, note that, for $1 \leq i \leq n$, $w_i = w_1^{(i-1)}$ and (4) implies that $z_i = z_1^{(i-1)}$, where w_i and z_i indicate the ith component of x and z, respectively. Moreover, $w_1 = \phi(z_1)$ and for $1 < i \leq n$ there exist functions $F_{i-1} : \mathbb{R}^{i-1} \to \mathbb{R}$ such that

$$w_i = F_{i-1}(z_1, \ldots, z_{i-1}) + \phi^{\langle 1\rangle}(z_1)z_i. \quad (37)$$

In fact, (37) holds for $i = 2$ with $F_1 \equiv 0$ because $w_2 = dw_1/dt = \phi^{\langle 1\rangle}(z_1)z_2$, and it is easy to see that if (37) holds for $i < n - 1$ it holds for $i + 1$ as well, since

$$\begin{aligned} w_{i+1} &= \sum_{j=1}^{i-1} \frac{\partial F_{i-1}}{\partial z_j} z_{j+1} + \phi^{\langle 2\rangle}(z_1)z_2z_i + \phi^{\langle 1\rangle}(z_1)z_{i+1} \\ &= F_i(z_1, \ldots, z_i) + \phi^{\langle 1\rangle}(z_1)z_{i+1}. \end{aligned} \quad (38)$$

Consequently, the Jacobian

$$\frac{dw}{dz} = \begin{bmatrix} \phi^{\langle 1\rangle}(z_1) & 0 & \ldots & 0 \\ * & \phi^{\langle 1\rangle}(z_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & \phi^{\langle 1\rangle}(z_1) \end{bmatrix}. \quad (39)$$

is non-singular because (35) implies $\phi^{\langle 1\rangle}(z_1) = 1/\mu(z_1)$, and $\mu(z_1) \neq 0$ by $\mathcal{H}_{1.3}$. ■

Next theorem states the main results of this section.

Theorem 3.1: *Assume that system (15)–(16) with $p = q = 1$ satisfies $\mathcal{H}_{1.1}$–$\mathcal{H}_{1.3}$. If ϕ and $\overline{\Theta}$ are chosen as in (35) and (36), respectively, and $a(w) = L_f^n\bar{h}\left(\overline{\Theta}^{-1}(w)\right)$ is uniformly Lipschitz in the domain of interest, then (15)–(16) is locally diffeomorphic to*

$$\dot{w}(t) = A_b w(t) + B_b\,(a(w(t)) + u(t)) \quad (40)$$

$$\bar{y}(t) = C_b w(t - \delta(t)) \quad (41)$$

through the map $w = \overline{\Theta}(x)$. Moreover, if $w = \overline{\Theta}(x)$ is a global diffeomorphism, then (15)–(16) and (40)–(41) are globally diffeomorphic.

Proof: Since by $\mathcal{H}_{1.1}$ system (15)–(16) has relative degree n, the map definition in (36) implies that $w = [w_1, w_1^{(1)}, \ldots, w_1^{(n-1)}]^T$. The components of this vector obey the following equations:

$$w_1^{(i)} = L_f^i\bar{h}(x), \quad i = 0, \ldots, n-1 \quad (42)$$

$$w_1^{(n)} = L_f^n\bar{h}(x) + \phi^{\langle 1\rangle}(h(x))\mu(h(x))u. \quad (43)$$

Indeed, for $i = 1$,

$$\begin{aligned} w_1^{(1)} &= \frac{d\bar{h}}{dx}\dot{x} = \frac{d\bar{h}}{dx}(f + gu) = L_f\bar{h} + \frac{d\bar{h}}{dy}\frac{dh}{dx}gu \\ &= L_f\bar{h} + \phi^{\langle 1\rangle} \cdot L_g h \cdot u = L_f\bar{h}. \end{aligned} \quad (44)$$

By induction,

$$\begin{aligned} w_1^{(i+1)} &= \frac{d}{dt}L_f^i\bar{h} = \frac{d}{dx}L_f^i\bar{h}(f + gu) \\ &= L_f^{i+1}\bar{h} + \frac{d}{dx}L_f^i\bar{h} \cdot gu. \end{aligned} \quad (45)$$

Using (33) in Lemma 3.1,

$$\begin{aligned} &\frac{d}{dx}L_f^i\bar{h} \cdot gu \\ &= \sum_{j=1}^{i} \binom{i-1}{j-1} \frac{d}{dx}\left(L_f^{i-j}\phi^{\langle 1\rangle} \cdot L_f^j h\right) \cdot gu \\ &= \sum_{j=1}^{i} \binom{i-1}{j-1} \left(L_f^j h \frac{dL_f^{i-j}\phi^{\langle 1\rangle}}{dy}\frac{dh}{dx} \cdot gu + L_f^{i-j}\phi^{\langle 1\rangle}\frac{dL_f^j h}{dx} \cdot gu\right) \\ &= \sum_{j=1}^{i} \binom{i-1}{j-1} L_f^{i-j}\phi^{\langle 1\rangle} \cdot L_g L_f^j h \cdot u. \end{aligned} \quad (46)$$

From the relative degree hypothesis it follows that $L_g L_f^j h = 0$ for $j < n - 1$, thus (46) is always null except when $j = n - 1$, or, since $n > i \geq j$, $i = j$ and $i + 1 = n$, in which case we obtain (43) because $L_f^{i-j}\phi^{\langle 1\rangle} = \phi^{\langle 1\rangle}$ and $L_g L_f^{n-1}h(x) = \mu(h(x))$ by $\mathcal{H}_{1.3}$.

By taking into account the choice of ϕ in (35), it results that $\phi^{\langle 1\rangle}(h(x))\mu(h(x)) = 1$. Therefore, (43) becomes

$$w_1^{(n)} = L_f^n\bar{h}(x) + u. \quad (47)$$

The state equation (40) can be computed by applying the map $w = \overline{\Theta}(x)$ and using (42) and (47). The output equation (41) can be obtained from (30)–(31) and (36) as it follows:

$$\bar{y}(t) = \phi(y(t)) = \phi\left(h\left(x\left(t-\delta\left(t\right)\right)\right)\right) = \bar{h}(x(t-\delta(t))) \\ = w_1(t-\delta(t)) = C_b w(t-\delta(t)). \tag{48}$$

The proof is concluded considering that, under the theorem hypotheses, Lemma 3.2 implies that $w = \overline{\Theta}(x)$ is a local diffeomorphism. Obviously, the two systems are globally diffeomorphic if $w = \overline{\Theta}(x)$ is a global diffeomorphism. ∎

Remark 3.3: Theorem 3.1 states that, under the hypotheses $\mathcal{H}_{1.1}$–$\mathcal{H}_{1.3}$, once the output transformation ϕ defined in (35) is applied, system (15)–(16) satisfies the hypotheses of Theorem 2.1. Therefore, if $w = \overline{\Theta}(x)$ is a global diffeomorphism there exists a state observer $\hat{x}(t) = \Gamma(y(t-\delta(t)))$ such that system (15)–(16) is globally exponentially stabilised with arbitrary decay rate by the control law (19) with $b = 1$.

3.2 Delayed output feedback control for MIMO systems

The results of the previous section can be extended to MIMO systems under additional hypotheses. We consider the system (15)–(16) with $p = q$ and $h(x) = [h_1(x), \ldots, h_q(x)]^T$. We assume that such a system satisfies the following hypotheses:

$\mathcal{H}_{2.1}$: has observation relative degree $\bar{r}$ such that $r_1 + \ldots + r_q = n$;
$\mathcal{H}_{2.2}$: the observability map

$$z = \Theta(x) = [h_1(x),\ \ldots,\ L_f^{r_1-1}h_1(x),\ \ldots, \\ h_q(x),\ \ldots,\ L_f^{r_q-1}h_q(x)]^T \tag{49}$$

is a global diffeomorphism;
$\mathcal{H}_{2.3}$: there exist $\mu_i : \mathbb{R}^q \to \mathbb{R}$, $i = 1, 2, \ldots, q$, such that, for all $x \in \mathbb{R}^n$,

$$\begin{bmatrix} L_gL_f^{r_1-1}h_1(x) \\ L_gL_f^{r_2-1}h_2(x) \\ \vdots \\ L_gL_f^{r_q-1}h_q(x) \end{bmatrix} = \begin{bmatrix} \mu_1(h(x)) & 0 & \ldots & 0 \\ 0 & \mu_2(h(x)) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_q(h(x)) \end{bmatrix} \\ = R(h(x)), \quad \mu_i(h(x)) \neq 0. \tag{50}$$

In analogy with the scalar case, we prove the existence of an output post-processing, consisting of a smooth and invertible function $\phi : \mathbb{R}^q \to \mathbb{R}^q$, designed to satisfy the separation principle stated in Theorem 2.3 by replacing the original output y with

$$\bar{y} = \phi(y) = [\phi_1(y), \phi_2(y), \ldots, \phi_q(y)]^T. \tag{51}$$

The corresponding output function is

$$\bar{h}(x) = [\bar{h}_1(x), \bar{h}_2(x), \ldots, \bar{h}_q(x)]^T, \ \text{with} \ \bar{h}_i(x) = \phi_i(h(x)). \tag{52}$$

Different from the scalar case, to obtain separation, we need to introduce the following additive hypotheses:

$\mathcal{H}_{2.4}$: the observation relative degrees r_i of the outputs h_i are distinct and $r_1 < r_2 < \ldots < r_q$;
$\mathcal{H}_{2.5}$: for all $i = 2, \ldots, q$ and $j = 1, \ldots, i-1$, $\partial\mu_i/\partial h_j = 0$.

Then, we define the output transformations ϕ_i as

$$\phi_i(y) = \int \frac{1}{\mu_i(y_i, y_{i+1}, \ldots, y_q)} dy_i \tag{53}$$

and the corresponding observability map as

$$w = \overline{\Theta}(x) = [\bar{h}_1(x),\ \ldots,\ L_f^{r_1-1}\bar{h}_1(x),\ \ldots, \\ \bar{h}_q(x),\ \ldots,\ L_f^{r_q-1}\bar{h}_q(x)]^T. \tag{54}$$

A key property of $\overline{\Theta}$ is formalised by the following lemma.

Lemma 3.3: *If the MIMO system (15)–(16) with $p = q$ satisfies $\mathcal{H}_{2.1}$–$\mathcal{H}_{2.5}$ and ϕ_i, $i = 1, \ldots, q$ are chosen as in (53), then $w = \overline{\Theta}(x)$ defined in (54) is a local diffeomophism.*

The proof of Lemma 3.3 is provided in Appendix A. The main result of this section is provided in the next theorem.

Theorem 3.2: *Assume that the MIMO system (15)–(16) with $p = q$ satisfies $\mathcal{H}_{2.1}$–$\mathcal{H}_{2.5}$. If ϕ_i, $i = 1, \ldots, q$ and $\overline{\Theta}$ are chosen as in (53) and (54), respectively, and*

$$a(w) = \begin{bmatrix} L_f^{r_1}\bar{h}_1\left(\overline{\Theta}^{-1}(w)\right) \\ L_f^{r_2}\bar{h}_2\left(\overline{\Theta}^{-1}(w)\right) \\ \vdots \\ L_f^{r_p}\bar{h}_p\left(\overline{\Theta}^{-1}(w)\right) \end{bmatrix}, \tag{55}$$

is uniformly Lipschitz in the domain of interest, then (15)–(16) is locally diffeomorphic to

$$\dot{w}(t) = A_{\bar{r}}w(t) + B_{\bar{r}}\left(a\left(w\left(t\right)\right) + u\left(t\right)\right) \tag{56}$$

$$\bar{y}(t) = C_{\bar{r}}w(t-\delta(t)), \tag{57}$$

through the map $w = \overline{\Theta}(x)$ defined in (54). Moreover, if $w = \overline{\Theta}(x)$ is a global diffeomorphism, then (15)–(16) and (56)–(57) are globally diffeomorphic.

Proof: Taking into account the map definition (54) and the hypotheses $\mathcal{H}_{2.4}$–$\mathcal{H}_{2.5}$, system (15)–(16) can be rewritten in the w coordinates as

$$\dot{w}(t) = A_{\bar{r}}w(t) + B_{\bar{r}}\left(a\left(\overline{\Theta}^{-1}(w(t))\right) + \begin{bmatrix} \frac{\partial\phi_1}{\partial y_1}\Big|_{y=h(x(t))} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial\phi_q}{\partial y_q}\Big|_{y=h(x(t))} \end{bmatrix} R(h(x(t)))u(t)\right) \tag{58}$$

$$\bar{y}(t) = C_{\bar{r}}w(t-\delta(t)), \tag{59}$$

which is identical to (56)–(57) since from (53) it follows that $\frac{\partial\phi_i}{\partial y_i}\Big|_{y=h(x)} = \frac{1}{\mu_i(h(x))}$. Lemma 3.3 implies that, under hypotheses $\mathcal{H}_{2.1}$–$\mathcal{H}_{2.5}$, $w = \overline{\Theta}(x)$ is a local diffeomorphism and therefore system (15)–(16) is locally diffeomorphic to system (56)–(57). Obviously, the two systems are globally diffeomorphic if $w = \overline{\Theta}(x)$ is a global diffeomorphism. ■

Remark 3.4: Theorem 3.2 states that under the hypotheses $\mathcal{H}_{2.1}$–$\mathcal{H}_{2.5}$, once the output transformation ϕ defined in (53) is applied, system (15)–(16) satisfies the hypotheses of Theorem 2.3. Therefore, if $w = \overline{\Theta}(x)$ is a global diffeomorphism there exists a state observer $\hat{x}(t) = \Gamma(y(t-\delta(t)))$ such that system (15)–(16) is globally exponentially stabilised with arbitrary decay rate by the control law (27) with $R = I_q$.

In the more general case, when the hypothesis $\mathcal{H}_{2.4}$ is replaced by

$\mathcal{H}'_{2.4}$: the relative degrees r_i of the outputs h_i are partially ordered, $r_1 \leq r_2 \leq \ldots \leq r_q$,

the output transformation $\bar{y} = H(y)$ that makes constant the input gain $R(h(x))$ exists only if there is a solution to the partial differential equation,

$$\frac{dH}{dy}\begin{bmatrix} L_gL_f^{r_1-1}h_1(x) \\ L_gL_f^{r_2-1}h_2(x) \\ \vdots \\ L_gL_f^{r_q-1}h_q(x) \end{bmatrix} = R, \quad R \in \mathbb{R}^{q\times q}, \ \det(R) \neq 0. \tag{60}$$

3.3 Predictor-based state feedback for delayed input systems

Consider a nonlinear system of the form (20) with $p = 1$ (single-input case) and satisfying the following hypotheses:

$\mathcal{H}_{3.1}$: it has relative degree n with respect to some $\lambda(x) = x_\ell$, $\ell \in \{1, 2, \ldots, n\}$;

$\mathcal{H}_{3.2}$: the map

$$z = \Theta(x) = [\lambda(x),\ L_f\lambda(x), \ldots,\ L_f^{n-1}\lambda(x)]^T \tag{61}$$

is a global diffeomorphism;

$\mathcal{H}_{3.3}$: there exists $\mu : \mathbb{R} \to \mathbb{R}$ such $b(x) = L_gL_f^{n-1}\lambda(x) = \mu(\lambda(x)) = \mu(x_\ell) \neq 0$ for all $x \in \mathbb{R}^n$.

Remark 3.5: Systems that satisfy hypotheses $\mathcal{H}_{3.1}$–$\mathcal{H}_{3.2}$ are feedback linearisable. As stated in Remark 2.5, it is possible to design a predictor-based controller for such systems when $L_gL_f^{n-1}\lambda(x)$ is a non-null constant b. We replace here this hypothesis with the less restrictive requirement $\mathcal{H}_{3.3}$ where $L_gL_f^{n-1}\lambda(x)$ can be a function of x_ℓ.

In analogy with the case of delay-output systems considered in Section 2.2, we prove the existence of a change of coordinates such that the transformed system satisfies the hypotheses of Theorem 2.2. We introduce the transformation

$$\phi(x_\ell) = \int \frac{1}{\mu(x_\ell)}dx_\ell = \bar{h}(x). \tag{62}$$

The observability map with respect to $\lambda(x) = x_\ell$ is

$$w = \overline{\Theta}(x) = [\bar{h}(x), L_f\bar{h}(x),\ \ldots,\ L_f^{n-1}\bar{h}(x)]^T. \tag{63}$$

Remark 3.6: Note that with the choice (62) the function $\phi(x_\ell)$ can be inverted, thanks to the fact that $\mu(x_\ell) \neq 0$, as assumed in $\mathcal{H}_{2.3}$.

A key property of $\overline{\Theta}$ is formalised by the following lemma.

Lemma 3.4: *If system (20) with $p = 1$ satisfies $\mathcal{H}_{3.1}$–$\mathcal{H}_{3.3}$ and ϕ is chosen as in (62), then $w = \overline{\Theta}(x)$ is a local diffeomorphism.*

Proof: Since by $\mathcal{H}_{3.2}$ $z = \Theta(x)$ is a global diffeomorphism, the thesis is proved if we show that $w = \widetilde{\Theta}(z)$ is a local diffeomorphism, that is, $dw/dz \neq 0$. By repeating the same steps in the proof of Lemma 3.2, we obtain that the Jacobian

$$\frac{dw}{dz} = \begin{bmatrix} \phi^{\langle 1\rangle}(z_1) & 0 & \ldots & 0 \\ * & \phi^{\langle 1\rangle}(z_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & \phi^{\langle 1\rangle}(z_1) \end{bmatrix}, \tag{64}$$

is non-singular since (62) implies $\phi^{\langle 1\rangle}(z_1) = 1/\mu(z_1)$ and $\mu(z_1) \neq 0$ by $\mathcal{H}_{3.3}$. ■

Next theorem states the main results of this section.

Theorem 3.3: *Assume that system (20) with $p = 1$ satisfies $\mathcal{H}_{3.1}$–$\mathcal{H}_{3.3}$. If ϕ and $\overline{\Theta}$ are chosen as in (62) and (63),* *respectively, and* $a(w) = L_f^n \bar{h}\left(\bar{\Theta}^{-1}(w)\right)$ *is uniformly Lipschitz in the domain of interest, then* (20) *is locally diffeomorphic to*

$$\dot{w}(t) = A_b w(t) + B_b(a(w(t)) + u(t - \delta(t))) \quad (65)$$

through the map $w = \bar{\Theta}(x)$*. Moreover, if* $w = \bar{\Theta}(x)$ *is a global diffeomorphism, then* (20) *and* (65) *are globally diffeomorphic.*

Proof: Since by $\mathcal{H}_{3.1}$ system (20) has relative degree n with respect to $\lambda(x) = x_\ell$, the map definition in (36) implies that $w = [w_1, w_1^{(1)}, \ldots, w_1^{(n-1)}]^T$. The components of this vector obey the following equations:

$$w_1^{(i)} = L_f^i \bar{h}(x), \quad i = 0, \ldots, n-1 \quad (66)$$

$$w_1^{(n)} = L_f^n \bar{h}(x) + \phi^{\langle 1 \rangle}(x_\ell)\mu(x_\ell)u. \quad (67)$$

Last equations can be proved using the same steps of the proof of Theorem 3.1 by assuming $h(x) = \lambda(x) = x_\ell$. By taking into account the choice of ϕ in (20), it results that $\phi^{\langle 1 \rangle}(x_\ell)\mu(x_\ell) = 1$. Therefore, (43) becomes

$$w_1^{(n)} = L_f^n \bar{h}(x) + u. \quad (68)$$

The state equation (65) can be computed by applying the map $w = \bar{\Theta}(x)$ and using (66) and (68). The proof is concluded considering that, under the theorem hypotheses, Lemma 3.4 implies that $w = \bar{\Theta}(x)$ is a local diffeomorphism. Obviously, the two systems are globally diffeomorphic if $w = \bar{\Theta}(x)$ is a global diffeomorphism. ■

Remark 3.7: Theorem 3.3 states that under the hypotheses $\mathcal{H}_{3.1}$–$\mathcal{H}_{3.3}$, once the transformation ϕ defined in (62) is applied, system (20) satisfies the hypotheses of Theorem 2.2. Therefore, if $w = \bar{\Theta}(x)$ is a global diffeomorphism there exists a predictor $\xi(t) = \Gamma(x(t))$ of the transformed state $w(t + \delta(t))$, such that system (20) is globally exponentially stabilised with arbitrary decay rate by the control law (22) with $b = 1$.

3.4 Predictor-based state feedback for delayed multi-input systems

The results of the previous section can be extended to the multi-input case using an approach similar to the one introduced in Section 3.2. We consider a system of the form (20) with $p > 1$. To simplify the presentation in analogy with Section 3.2, we can assume to have as fictitious outputs $y_i = h_{\ell_i}(x) = x_{\ell_i}$, $\ell_i \in \{1, 2, \ldots, n\}$ a subset of p state variables, which satisfy the following assumptions:

$\mathcal{H}_{4.1}$: the observation relative degree $\bar{r}$ such that $r_1 + \cdots + r_p = n$;

$\mathcal{H}_{4.2}$: the observability map

$$z = \Theta(x) = [h_1(x), \ldots, L_f^{r_1-1}h_1(x), \ldots, h_p(x), \ldots, L_f^{r_p-1}h_p(x)]^T \quad (69)$$

is a global diffeomorphism;

$\mathcal{H}_{4.3}$: there exist $\mu_i : \mathbb{R}^p \to \mathbb{R}$, $i = 1, 2, \ldots, q$, such that, for all $x \in \mathbb{R}^n$,

$$\begin{bmatrix} L_g L_f^{r_1-1}h_1(x) \\ L_g L_f^{r_2-1}h_2(x) \\ \vdots \\ L_g L_f^{r_p-1}h_p(x) \end{bmatrix} = \begin{bmatrix} \mu_1(y) & 0 & \ldots & 0 \\ 0 & \mu_2(y) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_p(y) \end{bmatrix} = R(y), \quad \mu_i(y) \neq 0. \quad (70)$$

$\mathcal{H}_{4.4}$: the observation relative degrees r_i of the outputs h_i are distinct and $r_1 < r_2 < \ldots < r_p$;

$\mathcal{H}_{4.5}$: for all $i = 2, \ldots, p$ and $j = 1, \ldots, i-1$, $\partial \mu_i / \partial y_j = 0$.

We can therefore define the output(state) transformation

$$\bar{y} = \phi(y) = [\phi_1(y), \phi_2(y), \ldots, \phi_p(y)]^T = \bar{h}(x) \quad (71)$$

with ϕ_i defined as in (53) with $q = p$ and the corresponding observability map $w = \bar{\Theta}(x)$ as in (49).

It is therefore possible to get the next two results.

Lemma 3.5: *If the multi-input system* (20) *satisfies* $\mathcal{H}_{4.1}$–$\mathcal{H}_{4.5}$ *and* ϕ_i, $i = 1, \ldots, p$ *are chosen as in* (53) *with* $q = p$*, then* $w = \bar{\Theta}(x)$ *defined in* (54) *is a local diffeomophism.*

Proof: This lemma can be proved exactly as Lemma 3.3. ■

Theorem 3.4: *Assume that the multi-input system* (20) *satisfies* $\mathcal{H}_{4.1}$–$\mathcal{H}_{4.5}$*. If* ϕ_i, $i = 1, \ldots,$ *and* $\bar{\Theta}$ *are chosen as in* (53) *and* (54) *with* $q = p$*, respectively, and*

$$a(w) = \begin{bmatrix} L_f^{r_1}\bar{h}_1\left(\bar{\Theta}^{-1}(w)\right) \\ L_f^{r_2}\bar{h}_2\left(\bar{\Theta}^{-1}(w)\right) \\ \vdots \\ L_f^{r_p}\bar{h}_p\left(\bar{\Theta}^{-1}(w)\right) \end{bmatrix}, \quad (72)$$

is uniformly Lipschitz in the domain of interest, then (20) *is locally diffeomorphic to*

$$\dot{w}(t) = A_{\bar{r}} w(t) + B_{\bar{r}}\left(a(w(t)) + u(t - \delta(t))\right) \quad (73)$$

through the map $w = \overline{\Theta}(x)$. Moreover, if $w = \overline{\Theta}(x)$ is a global diffeomorphism, then (20) and (73) are globally diffeomorphic.

Proof: This theorem can be proved as Theorem 3.3. ■

Remark 3.8: Theorem 3.4 states that under the hypotheses $\mathcal{H}_{4.1}$–$\mathcal{H}_{4.5}$, once the transformation ϕ defined in (53) is applied, the multi-input system (20) satisfies the hypotheses of Theorem 2.4. Therefore, if $w = \overline{\Theta}(x)$ is a global diffeomorphism there exists a predictor $\xi(t) = \Gamma(x(t))$ of the transformed state $w(t + \delta(t))$, such that system (20) is globally exponentially stabilised with arbitrary decay rate by the control law (29) with $R = I_p$.

3.5 Predictor-based output feedback for input and output delayed systems

Combining the above output feedback controller predictor/controller schemes with the results available in Cacace et al. (2014) it is easy to prove the following.

Proposition 3.1: *Consider the nonlinear SISO system*

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t - \delta_i(t)) \tag{74}$$

$$y(t) = h(x(t - \delta_o(t))), \tag{75}$$

where the time-varying input and output delay functions $\delta_i(t)$ and $\delta_o(t)$ are known and have arbitrary bounds. If (74)–(75) has relative degree n and there exist a function $\mu : \mathbb{R} \to \mathbb{R}$ such that

$$L_g L_f^{n-1}(h(x)) = \mu(h(x)) \neq 0, \tag{76}$$

then there exists an exponentially stabilising output feedback controller with arbitrary exponential decay rate.

In analogy with Section 3.2, the result can be extended to provide sufficient conditions for the MIMO case.

4. Examples

4.1 Output feedback control from sampled scalar measurements

We consider the model proposed in Hahnfeldt, Panigrahy, Folkman, and Hlatky (1999) for the growth of a solid tumor under anti-angiogenic administration

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = f(x) + Bu(t), \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$f(x) = \begin{bmatrix} -\lambda x_1 \ln\left(\frac{x_1}{x_2}\right) \\ bx_1 - (m + dx_1^{2/3})x_2 - cx_2x_3 \\ -\eta x_3 \end{bmatrix} \tag{77}$$

where x_1 and x_2 denote the tumor volume and the carrying capacity of the vasculature, respectively, x_3 is the anti-angiogenic concentration, and u stands for the anti-angiogenic drug administration rate, supposed to be delivered continuously. The rational behind (77) is that the tumor size is limited by the extent of the vasculature, that can be controlled by means of an anti-angiogenic drug. The domain of interest is the positive octant of $\mathbb{R}^3$, and u is positive. The task of the control is to bring the tumor size from its initial value $x_1(0)$ to a reference value $\bar{x}_1$. To this aim, it is necessary to design an output feedback controller based on the measured variable x_1. In practice, the measurement of x_1 cannot be taken continuously and the resulting sampled output is

$$y(kT) = h(x(kT)) = x_1(kT) = x_1(t - \delta(t)), \tag{78}$$

where $t \geq 0$, T is the sampling time, $k = \lfloor t/T \rfloor$ is the largest integer less than or equal to t/T and the sampling process is modelled as the time-varying delay $\delta(t) = t - kT$, for $t \in [kT, (k+1)T)$. Consequently, $\delta(t) \in [0, T)$. This approach allows to transform the mixed continuous-discrete system (77)–(78) into a continuous-time system with time-varying non continuous delays, as originally suggested in Mikheev, Sobolev, and Fridman (1988). It is easy to verify that (77) has relative degree 3 and it is feedback linearisable. The nonlinear map

$$z = \Theta(x) = \begin{bmatrix} x_1 - \bar{x}_1 \\ L_f h(x) \\ L_f^2 h(x) \end{bmatrix} \tag{79}$$

has a non-singular Jacobian in the domain of interest, thus it is a local diffeomorphism. In the z coordinates, the system is in the form (5) with $L_g L_f^2 h(x) = -\lambda c x_1 \neq 0$. The system therefore satisfies the separation condition when the output is $x_1(t)$, *i.e.* there is no measurement delay, and in this case the estimation error of a high-gain observer of (77)–(78) does not depend on the feedback gain. However, when the output is sampled as in (78) $x_1(kT)$ is the only information available and the estimation error of the observer for delayed measurements depends on the control gain K. Thus, the asymptotic convergence to $\bar{x}_1$ of the output feedback control is no longer globally guaranteed for an independent choice of K and L (the observer gain), as discussed in Section 2. In order to restore the separation principle, we can resort to the result of Section 3.1, since $L_g L_f^2 h(x) = -\lambda c x_1 = \mu(h(x))$ depends only on the output variable $h(x) = x_1$, thus satisfying $\mathcal{H}_{1.3}$. Therefore, by defining

$$\bar{h}(x) = \int_{\bar{x}_1}^{x_1} \frac{1}{-\lambda c s} ds = \frac{1}{\lambda c} \ln\left(\frac{\bar{x}_1}{x_1}\right), \tag{80}$$

$$w = \bar{\Theta}(x) = \begin{bmatrix} \bar{h}(x) \\ L_f\bar{h}(x) \\ L_f^2\bar{h}(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda c}\ln\left(\frac{\bar{x}_1}{x_1}\right) \\ \frac{1}{c}\ln\left(\frac{x_1}{x_2}\right) \\ -\frac{\lambda}{c}\ln\left(\frac{x_1}{x_2}\right) - \frac{bx_1}{cx_2} + \frac{1}{c}\left(m + dx_1^{\frac{2}{3}}\right) + x_3 \end{bmatrix} \quad (81)$$

We have that $w = \bar{\Theta}(x)$ is still a local diffeomorphism with Jacobian

$$\bar{Q}(x) = \frac{d\bar{\Theta}}{dx} = \begin{bmatrix} \frac{-1}{\lambda c x_1} & 0 & 0 \\ \frac{1}{cx_1} & \frac{-1}{cx_2} & 0 \\ \frac{1}{c}\left(\frac{2}{3}dx_1^{-\frac{1}{3}} - \frac{\lambda}{x_1} - \frac{b}{x_2}\right) & \frac{\lambda x_2 + bx_1}{cx_2^2} & 1 \end{bmatrix}, \quad (82)$$

which is non-singular in the same domain of interest as the original Jacobian $d\Theta/dx$. From (81) and (77) it follows that $L_gL_f^2\bar{h}(x) = 1$. System (77)–(78) becomes in the new coordinates

$$\dot{w} = A_b w(t) + B_b(L_f^3\bar{h}(x(t)) + u(t)) \quad (83)$$

$$\bar{y}(kT) = \bar{h}(x(kT)) = \frac{1}{\lambda c}\ln\left(\frac{\bar{x}_1}{x_1(kT)}\right) = \frac{1}{\lambda c}\left(\ln(\bar{x}_1) - \ln(y(kT))\right). \quad (84)$$

By applying Theorem 3.1 we have that the output feedback controller composed by the observer for delayed measurements and the linearising feedback computed on the observer estimate is globally stabilising for a bounded value of T and suitable and independent choices of the observer gain L and K, detailed below. For completeness, we report the equations of the output feedback controller when the observer equation is written in the original x coordinates by using

$$\dot{x} = (\bar{Q}(x))^{-1}\dot{w}, \quad (85)$$

that avoids the need of explicitly inverting the nonlinear map $w = \bar{\Theta}(x)$. The resulting output feedback controller is

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + Bu(t) + \frac{1}{\lambda c}(\bar{Q}(\hat{x}))^{-1}L(t)\ln\left(\frac{\hat{x}_1(kT)}{y(kT)}\right), \quad (86)$$

$$u(t) = -K\bar{\Theta}(\hat{x}(t)) - L_f^3\bar{h}(\hat{x}(t)). \quad (87)$$

In order to have global exponential stability at $w = 0$, that is, $x_1 = \bar{x}_1$, the control gain K is designed to make $A_b - B_bK$ Hurwitz. Following the approach in Cacace, Germani, and Manes (2012), the time-varying gain $L(t)$ is computed as $L(t) = L_0e^{-\alpha\delta(t)}$, where $\delta(t) = t - kT$ is the delay function corresponding to sampling, $\alpha > 0$ is a design parameter and L_0 is such that $\rho(A_b - L_0C_b)) < -\alpha$. Cacace et al. (2015) provides sufficient conditions for the exponential convergence to zero of the observer error with rate α, expressed as a bound on the sampling interval T. The observer state is guaranteed to converge to the state of the system when the value of T is 'not too large', with a bound that depends on α and on the nonlinearities of the system. When T satisfies this condition, the output feedback controller (86)–(87) is globally exponentially stabilising to $\bar{x}$. The exponential rate of convergence to zero of $\|w(t)\|$, and consequently of $\|\log(\bar{x}_1/x_1)\|$, is thus given by $\min\{\alpha, -\rho(A_b - B_bK)\}$.

In order to validate our conclusions, Figures 1 and 2 plot the behaviour of the closed-loop system when $\lambda = 0.291$ day^{-1}, $b = 5.85$ day^{-1}, $d = 0.00873$ day^{-1} mm^{-2}, $c = 0.66$ day^{-1}, $\eta = 1.7$ day^{-1}. $m = 0$ day^{-1}. The initial state of the system and the observer in the plots is $x(0) = [200, 625, 0]^T$, $\hat{x}(0) = [200,\ 300,\ 0.1]^T$. The set-point for the controller is $\bar{x}_1 = 50$. The time scale is in days, and $T = 0.5$ days, that is, the controller uses two measurements per day. The gains L_0 and K in (86)–(87) are designed to assign to $A_b - L_0C_b$ and $A_b - B_bK$ the eigenvalues $\{-1.5, -1.55, -1.6\}$ and $\{-0.7, -0.75, -0.8\}$ respectively, and $L(t) = e^{-2\delta(t)}L_0$, that is, $\alpha = 2$. The first part of the left plot of Figure 1 shows how fast the system would diverge in open-loop without control. It is also remarkable that the observer estimate converges by using only 4–5 samples of the output, that is, in about 2 days. Figure 2 confirms the exponential convergence to zero with the desired rate, $\alpha = 2$ for the observer (left-hand plot) and $-\rho(A_b - L_0C_b) = 0.7$ for the controlled state in w coordinates (right-hand plot). Note that this implies the exponential convergence to zero of $\log(\bar{x}_1/x_1(t))$, and therefore of the relative displacement $1 - \bar{x}_1/x_1(t)$ with the same rate.

4.2 Predictor-based state feedback control of a delayed multi-input system

Consider the following system:

$$\dot{x}_1(t) = x_1(t)x_3(t) + x_1(t)x_2(t)u_1(t-\delta) \quad (88)$$

$$\dot{x}_2(t) = x_2^2(t) + x_3(t) \quad (89)$$

$$\dot{x}_3(t) = x_2(t)x_3(t) + x_2(t)u_2(t-\delta) \quad (90)$$

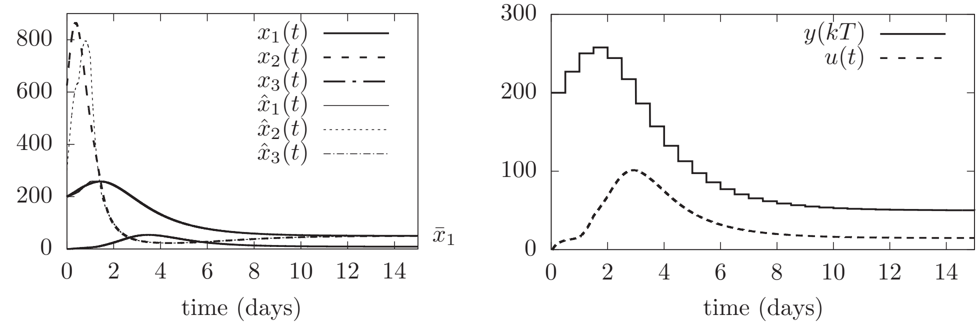

Figure 1. System and observer state (left); sampled output with $T = 0.5$ days and input generated by the output feedback controller (right).

where the delay δ is known and arbitrarily bounded. This system admits the set of equilibrium points $\mathcal{M} = \{x : x = [\alpha \;\; \beta \;\; -\beta^2]^T \in \mathbb{R}^3,\ \alpha, \beta \in \mathbb{R}\}$ obtained for the steady-state input $\bar{u} = [\beta \;\; \beta^2]^T$. The problem is to asymptotically stabilise any equilibrium point $\bar{x} \in \mathcal{M}$ with $\alpha, \beta \neq 0$ supposing the availability of the state vector $x(t)$ at time t.

The idea is to use the predictor-based control proposed in Cacace et al. (2016b). To this end, we follow the approach described in Section 3.4. It is easy to verify that system (88)–(90) satisfies $\mathcal{H}_{4.1}$–$\mathcal{H}_{4.3}$ and locally around $\bar{x}$ it is diffeomorphic to the feedback linearisable system obtained by choosing as fictitious output the function

$$y = h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{91}$$

In fact, the system has relative degree $r_1 = 1$ with respect to h_1 and $r_2 = 2$ with respect to h_2. Therefore $\mathcal{H}_{4.1}$ is satisfied since $r_1 + r_2 = 3$. Moreover,

$$R(y) = \begin{bmatrix} L_g L_f^{r_1-1} h_1 \\ L_g L_f^{r_2-1} h_2 \end{bmatrix} = \begin{bmatrix} \mu_1(y) & 0 \\ 0 & \mu_2(y) \end{bmatrix} = \begin{bmatrix} y_1 y_2 & 0 \\ 0 & y_2 \end{bmatrix}, \tag{92}$$

is non-singular if $x_1 \neq 0$ and $x_2 \neq 0$, as required by $\mathcal{H}_{4.3}$, and the nonlinear map

$$z = \Theta(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 + x_3 \end{bmatrix} \tag{93}$$

is a global diffeomophism, as required by $\mathcal{H}_{4.2}$. Under these conditions, system (88)–(90) is globally diffeomorphic to

$$\dot{z}(t) = A_{\bar{r}} z(t) + B_{\bar{r}} \left(a_z \left(\Theta^{-1}(z(t)) \right) + R(y(t)) u(t - \delta) \right) \tag{94}$$

with

$$a_z(x) = \begin{bmatrix} x_1 x_3 \\ x_2 x_3 \end{bmatrix}. \tag{96}$$

The predictor-based controller of Theorem 2.4 requires $R(y)$ to be a constant matrix. Since $\mathcal{H}_{4.4}$ and $\mathcal{H}_{4.5}$ are satisfied, we can apply Theorem 3.4 to obtain $R(y) = I_2$ by means of the transformation (53), which defines a new

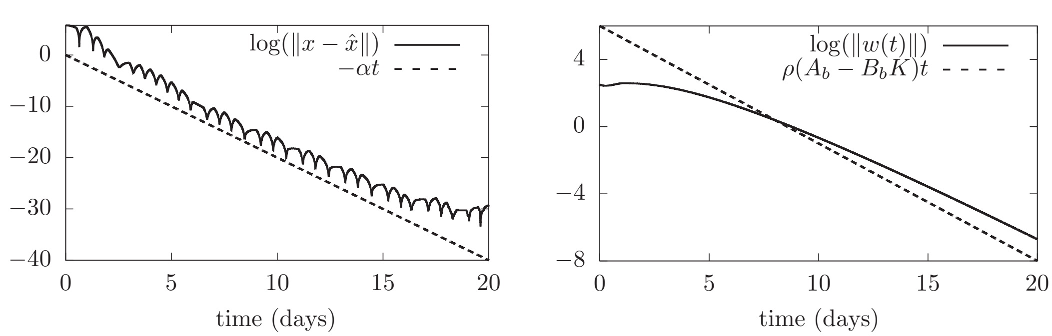

Figure 2. The norm $\|x(t) - \hat{x}(t)\|$ converges to zero with an exponential rate equal to the parameter α set for the observer (left, logarithmic scale). The norm $\|w(t)\|$ of the system in the new coordinates converges to zero with an exponential rate given by $\rho(A_b - B_b K)$ (right, logarithmic scale).

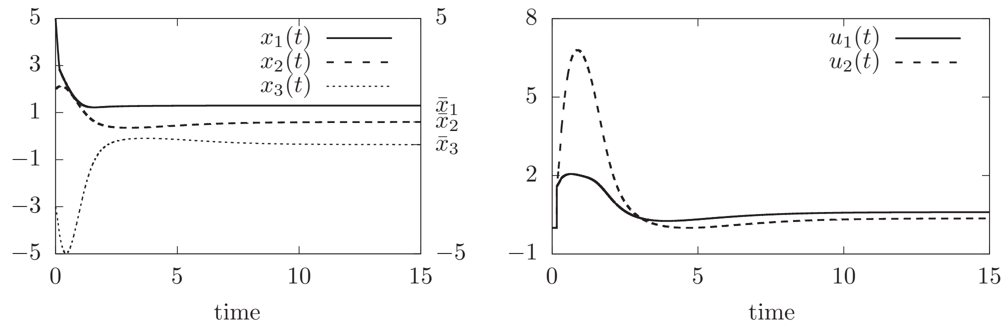

Figure 3. System state (left); control input generated by the predictor-based controller (right).

fictitious output

$$\bar{h}_1(x) = \int_{\bar{x}_1}^{x_1} \frac{1}{s y_2} ds = \frac{1}{x_2} \ln\left(\frac{x_1}{\bar{x}_1}\right) \quad (97)$$

$$\bar{h}_2(x) = \int_{\bar{x}_2}^{x_2} \frac{1}{s} ds = \ln\left(\frac{x_2}{\bar{x}_2}\right) \quad (98)$$

and the map

$$w = \bar{\Theta}(x) = \begin{bmatrix} \bar{h}_1(x) \\ \bar{h}_2(x) \\ L_f \bar{h}_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{x_2} \ln\left(\frac{x_1}{\bar{x}_1}\right) \\ \ln\left(\frac{x_2}{\bar{x}_2}\right) \\ x_2 + \frac{x_3}{x_2} \end{bmatrix} \quad (99)$$

which is a local diffeomorphism around $\bar{x} \neq 0$. In the coordinates $w = \bar{\Theta}(x)$, system (88)–(90) assumes the form

$$\dot{w}(t) = A_{\bar{r}} w(t) + B_{\bar{r}} \left(a(w(t)) + u(t-\delta)\right),$$
$$a_w(w) = \bar{a}\left(\bar{\Theta}^{-1}(w)\right) \quad (100)$$

with

$$\bar{a}(x) = \begin{bmatrix} \ln\left(\frac{\bar{x}_1}{x_1}\right)\left(1 + \frac{x_3}{x_2^2}\right) + \frac{x_3}{x_2} \\ x_2^2 - \frac{x_3^2}{x_2^2} + x_3 \end{bmatrix},$$
$$\bar{\Theta}^{-1}(w) = \begin{bmatrix} \bar{x}_1 e^{w_1 \bar{x}_2 e^{w_2}} \\ \bar{x}_2 e^{w_2} \\ \bar{x}_2 e^{w_2}(w_3 - \bar{x}_2 e^{w_2}), \end{bmatrix}. \quad (101)$$

from which

$$a_w(w) = \begin{bmatrix} -w_1 w_3 + w_3 - \bar{x}_2 e^{w_2} \\ 3 w_3 \bar{x}_2 e^{w_2} - w_3^2 - \bar{x}_2^2 e^{2 w_2} \end{bmatrix} \quad (102)$$

Finally, the equilibrium point $\bar{x}$ can be exponentially stabilised using the control law (29). The state feedback

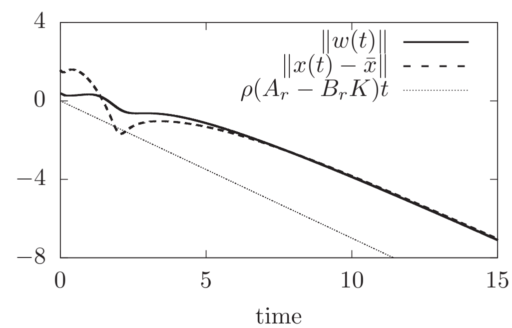

Figure 4. The norms $\|w(t)\|$ and $\|x(t) - \bar{x}\|$ converge to zero with an exponential rate given by $\rho(A_{\bar{r}} - B_{\bar{r}}K)$ (logarithmic scale).

controller for delayed input assumes the form

$$u(t) = -K\xi(t) - a_w(\xi(t)), \quad (103)$$
$$\xi(t) = \Gamma(x(t)) = e^{\bar{A}\delta} \bar{\Theta}(x(t)), \; \bar{A} = A_{\bar{r}} - B_{\bar{r}} K, \quad (104)$$
$$A_{\bar{r}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{\bar{r}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (105)$$

where $\xi(t)$ is the prediction of $w(t+\delta)$ from $x(t)$, designed according to Cacace et al. (2016b). The predictor $\xi(t)$ can also be obtained by integrating the following delay differential equation (DDE):

$$\dot{\xi}(t) = \bar{A}\xi(t) + e^{\bar{A}\delta} B_{\bar{r}} (K(w(t) - \xi(t-\delta))$$
$$+ a(w(t)) - a(\xi(t-\delta))), \quad (106)$$

(see Cacace et al., 2016b for the equivalence between (104) and (106)), and in this form it is possible to design a chain of predictors for delay functions with arbitrary bounds.

Figures 3 and 4 show the behaviour of the controller for system (88)–(90) when $\delta = 0.16$, $x_0^T = [5,\ 2,\ -3]$, $\bar{x}^T = [1.3, 0.6, -0.36]$ and K such to assign the eigenvalues $\{-0.7, -0.75, -0.8\}$ to $\bar{A}$. Note that, in Figure 3 (right) the plot refers to the time at which the plant receives the input, consequently in $[0, \delta]$ no control is applied. The desired rate $\rho(\bar{A})$ of exponential convergence to zero of $x(t) - \bar{x}$ is achieved at this delay as shown in Figure 4. The small value of the delay δ used in these simulations is due to the fact that the maximum delay value for which the predictor (104) can be used is about 0.18. The small delay bound is due to the strong nonlinearities of the system. To cope with larger delays is possible to resort to a chain of predictors (see Cacace et al., 2016b), that would in principle allow to cope with any bounded delay. In view of this, system (88)–(90) can be globally stabilised for *any* bounded input delay by using the proposed change of coordinates (99) and the appropriate predictor, which in general may be more complex than (106).

5. Conclusions

Strict-feedback nonlinear systems can be stabilised by output feedback and state feedback under arbitrarily large input/output delays. This paper proves that such class is equivalent, under a change of coordinates, to the wider class of feedback linearisable systems with multiplicative input when the multiplicative term is a function of measured variables. We have applied this result to one specific problem, the design of output feedback controllers and predictor-based controllers for nonlinear systems with input and output delays. However, this result may have other applications, for example it implies the existence of *white-noise solutions* for a class of feedback linearisable stochastic systems (Cacace et al., 2016a). The extension of this approach to a wider class of functions deserves further studies.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

Ahmed-Ali, T., Cherrier, E., & Lamnabhi-Lagarrigue, F. (2012). Cascade high gain predictors for a class of nonlinear systems. *IEEE Transactions on Automatic Control, 57*(1), 221–226.

Bekiaris-Liberis, N., & Krstic, M. (2012). Compensation of time-varying input and state delays for nonlinear systems. *Journal of Dynamic Systems, Measurement, and Control, 134*(1), 011009.

Bekiaris-Liberis, N., & Krstic, M. (2013a). Compensation of state-dependent input delay for nonlinear systems. *IEEE Transactions on Automatic Control, 58*(2), 275–289.

Bekiaris-Liberis, N., & Krstic, M. (2013b). *Nonlinear control under nonconstant delays. Advanced in Design and Control*. Philadelphia, PA: SIAM.

Cacace, F., Conte, F., Germani, A., & Palombo, G. (2016). White noise solution for nonlinear stochastic systems. In *Proceedings of 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Monterey, US.

Cacace, F., Conte, F., Germani, A., & Pepe, P. (2016). Stabilization of strict-feedback nonlinear systems with input delay using closed-loop predictors. *International Journal of Robust and Nonlinear Control,* 26(16), 3524–3540.

Cacace, F., Germani, A., & Manes, C. (2012). An exponential observer with delay-dependent gain for a class of nonlinear systems with time-varying measurement delay. In *Proceedings of the 51st IEEE Conference on Decision and Control CDC* (pp. 2364–2369), Maui, Hawaii.

Cacace, F., Germani, A., & Manes, C. (2014). A chain observer for nonlinear systems with multiple time-varying measurement delays. *SIAM Journal Control and Optimization, 52*(3), 1862–1885.

Cacace, F., Germani, A., & Manes, C. (2015). State estimation and control of nonlinear systems with large and variable measurement delays. In I. Karafyllis M. Malisoff F. Mazenc P. Pierdomenico (Eds.), *Recent results on nonlinear delay control systems* (Ch. 5, pp. 95–112). Heidelberg: Springer International Publishing.

Ciccarella, G., Dalla Mora, M., & Germani, A. (1993). A Luemberger-like observer for nonlinear systems. *International Journal of Control, 57*(3), 537–556.

Dalla Mora, M., Germani, A., & Manes, C. (2000). Design of state observers from a drift-observability property. *IEEE Transactions on Automatic Control, 45*(8), 1536–1540.

Datko, R. (1988). Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM Journal Control and Optimization, 26*, 697–713.

Fridman, E. (2014). *Introduction to time-delay systems*. Boston, MA: Birkhäuser.

Hahnfeldt, P., Panigrahy, D., Folkman, J., & Hlatky, L. (1999). Tumor development under angiogenic signaling: A dynamical theory of tumor growth, treatment response, and postvascular dormancy. *Cancer Research, 59*, 4770–4775.

Henson, M. A., & Seborg, D.E. (1994). Time delay compensation for nonlinear processes. *Industrial & Engineering Chemistry Research, 33*(6), 1493–1500.

Hu, Q., & Rangaiah, G.P. (1999). A time delay compensation strategy for uncertain single-input single-output nonlinear processes. *Industrial & Engineering Chemistry Research, 38*(11), 4309–4316.

Isidori, A. (1995). *Nonlinear control systems*. London: Springer-Verlag.

Jankovic, M. (2001). Control Lyapunov– Razumikhin functions and robust stabilization of time delay systems. *IEEE Transactions on Automatic Control, 46*(7), 1048–1060.

Karafyllis, I. (2011). Stabilization by means of approximate predictors for systems with delayed input. *SIAM Journal on Control and Optimization, 49*(3), 1100–1123.

Karafyllis, I., & Krstic, M. (2012). Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Transactions on Automatic Control, 57*(5), 1141–1154.

Karafyllis, I., & Krstic, M. (2013). Stabilization of nonlinear delay systems using approximate predictors and high-gain observers. *Automatica, 49*(12), 3623–3631.

Karafyllis, I., Krstic, M., Ahmed-Ali, T., & Lamnabhi-Lagarrigue, F. (2014). Global stabilisation of nonlinear delay systems with a compact absorbing set. *International Journal of Control, 87*(5), 1010–1027.
Khalil, H. K. (2015). *Nonlinear control*. Pearson Education.
Kravaris, C., & Wright, R.A. (1989). Deadtime compensation for nonlinear processes. *AIChE Journal, 35*(9), 1535–1542.
Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Boston, MA: Birkhäuser.
Krstic, M. (2010a). Compensation of infinite-dimensional actuator and sensor dynamics. *IEEE Control Systems, 30*(1), 22–41.
Krstic, M. (2010b). Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Transactions on Automatic Control, 55*(2), 287–303.
Lei, J., & Khalil, H.K. (2016a). High-gain-predictor-based output feedback control for time-delay nonlinear systems. *Automatica, 71*, 324–333.
Lei, J., & Khalil, H.K. (2016b). Feedback linearization for nonlinear systems with time-varying input and output delays by using high-gain predictors. *IEEE Transactions on Automatic Control, 61*(8), 2262–2268.
Mazenc, F., & Bliman, P.-A. (2006). Backstepping design for time-delay nonlinear systems. *IEEE Transactions on Automatic Control, 51*(1), 149–154.
Mazenc, F., & Malisoff, M. (2016). Stabilization of nonlinear time-varying systems through a new prediction based approach. IEEE Transactions on Automatic Control. doi:10.1109/TAC.2016.2600500.
Mazenc, F., Mondie, S., & Francisco, R. (2004). Global asymptotic stabilization of feedforward systems with delay in the input. *IEEE Transactions on Automatic Control, 49*(5), 844–850.
Mikheev, Y., Sobolev, V., & Fridman, E. (1988). Asymptotic analysis of digital control systems. *Automatic Remote Control, 49*, 1175–1180.
Oguchi, T., Watanabe, A., & Nakamizo, T. (2002). Input-output linearization of retarded non-linear systems by using an extension of Lie derivative. *International Journal of Control, 75*(8), 582–590.
Ortega, R., & Lozano, V. (1988). Globally stable adaptive controller for systems with delay. *International Journal of Control, 47*(1), 17–23.
Zhong, Q.C. (2006). *Robust control of time-delay systems*. New York, NY: Springer.

Appendix Proof of Lemma 3.3

Since by $\mathcal{H}_{2.2}$, $z = \Theta(x)$ defined in (49) is a global diffeomorphism, the thesis is proved if we show that $w = \widetilde{\Theta}(z)$ is a local diffeomorphism, that is, $dw/dz \neq 0$. To this end, let us reorder the maps $z = \Theta(x)$ and $w = \overline{\Theta}(x)$ as it follows:

$$w = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_n \end{bmatrix}, \quad z = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \vdots \\ \tilde{z}_n, \end{bmatrix}, \tag{A1}$$

with

$$\tilde{w}_1 = \begin{bmatrix} \bar{h}_1(x) \\ \bar{h}_2(x) \\ \vdots \\ \bar{h}_q(x) \end{bmatrix}, \quad \tilde{w}_2 = \begin{bmatrix} L_f\bar{h}_1(x) \\ L_f\bar{h}_2(x) \\ \vdots \\ L_f\bar{h}_q(x) \end{bmatrix}, \ldots,$$

$$\tilde{w}_{r_1} = \begin{bmatrix} L_f^{r_1-1}\bar{h}_1(x) \\ L_f^{r_1-1}\bar{h}_2(x) \\ \vdots \\ L_f^{r_1-1}\bar{h}_q(x) \end{bmatrix}, \tag{A2}$$

$$\tilde{w}_{r_1+1} = \begin{bmatrix} L_f^{r_1}\bar{h}_2(x) \\ L_f^{r_1}\bar{h}_3(x) \\ \vdots \\ L_f^{r_1}\bar{h}_q(x) \end{bmatrix}, \quad \tilde{w}_{r_1+2} = \begin{bmatrix} L_f^{r_1+1}\bar{h}_2(x) \\ L_f^{r_1+1}\bar{h}_3(x) \\ \vdots \\ L_f^{r_1+1}\bar{h}_q(x) \end{bmatrix}, \ldots,$$

$$\tilde{w}_{r_1+r_2} = \begin{bmatrix} L_f^{r_2}\bar{h}_2(x) \\ L_f^{r_2}\bar{h}_3(x) \\ \vdots \\ L_f^{r_2}\bar{h}_q(x) \end{bmatrix}, \ldots \tag{A3}$$

$$\ldots, \tilde{w}_{r_1+\cdots+r_{q-1}} = L_f^{r_q}\bar{h}_q(x), \quad \tilde{w}_{r_1+\cdots+r_{q-1}+1} = L_f^{r_q+1}\bar{h}_q(x), \ldots, \quad \tilde{w}_n = L_f^{r_q-1}\bar{h}_q(x) \tag{A4}$$

and

$$\tilde{z}_1 = \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_q(x) \end{bmatrix}, \quad \tilde{z}_2 = \begin{bmatrix} L_fh_1(x) \\ L_fh_2(x) \\ \vdots \\ L_fh_q(x) \end{bmatrix}, \ldots,$$

$$\tilde{z}_{r_1} = \begin{bmatrix} L_f^{r_1-1}h_1(x) \\ L_f^{r_1-1}h_2(x) \\ \vdots \\ L_f^{r_1-1}h_q(x) \end{bmatrix}, \tag{A5}$$

$$\tilde{z}_{r_1+1} = \begin{bmatrix} L_f^{r_1}h_2(x) \\ L_f^{r_1}h_3(x) \\ \vdots \\ L_f^{r_1}h_q(x) \end{bmatrix}, \quad \tilde{z}_{r_1+2} = \begin{bmatrix} L_f^{r_1+1}h_2(x) \\ L_f^{r_1+1}h_3(x) \\ \vdots \\ L_f^{r_1+1}h_q(x) \end{bmatrix}, \ldots,$$

$$\tilde{z}_{r_1+r_2} = \begin{bmatrix} L_f^{r_2}h_2(x) \\ L_f^{r_2}h_3(x) \\ \vdots \\ L_f^{r_2}h_q(x) \end{bmatrix}, \ldots \tag{A6}$$

$$\ldots, \tilde{z}_{r_1+\cdots+r_{q-1}} = L_f^{r_q}h_q(x), \quad \tilde{z}_{r_1+\cdots+r_{q-1}+1} = L_f^{r_q+1}h_q(x), \ldots, \quad \tilde{z}_n = L_f^{r_q-1}h_q(x) \tag{A7}$$

We also define $\tilde{y}_i$ and $\tilde{\phi}_i$ as it follows:

$$\begin{aligned}
\tilde{y}_i &= [y_1, y_2, \cdots, y_q]^T, & \tilde{\phi}_i &= [\phi_1, \phi_2, \cdots, \phi_q]^T, \\
& & &\text{for } i = 1, \ldots, r_1, \\
\tilde{y}_i &= [y_2, y_3, \cdots, y_q]^T, & \tilde{\phi}_i &= [\phi_2, \phi_3, \cdots, \phi_q]^T, \\
& & &\text{for } i = r_1 + 1, \ldots, r_1 + r_2, \\
\tilde{y}_i &= [y_2, y_4, \cdots, y_q]^T, & \tilde{\phi}_i &= [\phi_3, \phi_4, \cdots, \phi_q]^T, \\
& & &\text{for } i = r_1 + r_2 + 1, \ldots, \\
& & &r_1 + r_2 + r_3, \\
&\vdots & &\vdots \\
\tilde{y}_i &= y_q, & \tilde{\phi}_i &= \phi_q, \\
& & &\text{for } i = r_1 + \cdots \\
& & &+ r_{q-1} + 1, \ldots, n.
\end{aligned} \qquad \text{(A8)}$$

Note now that $\tilde{w}_1 = \phi(\tilde{z}_1)$. Moreover, there exist functions H_{i-1} such that

$$\tilde{w}_i = H_{i-1}(\tilde{z}_1, \ldots, \tilde{z}_{i-1}) + \left.\frac{d\tilde{\phi}_i}{d\tilde{y}_i}\right|_{y=\tilde{z}_1} \tilde{z}_i. \qquad \text{(A9)}$$

Consequently, the Jacobian

$$\frac{dw}{dz} = \begin{bmatrix} \left.\frac{d\tilde{\phi}_1}{d\tilde{y}_1}\right|_{y=\tilde{z}_1} & 0 & \cdots & 0 \\ * & \left.\frac{d\tilde{\phi}_2}{d\tilde{y}_2}\right|_{y=\tilde{z}_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \left.\frac{d\tilde{\phi}_n}{d\tilde{y}_n}\right|_{y=\tilde{z}_1} \end{bmatrix}. \qquad \text{(A10)}$$

is non-singular because (53) and $\mathcal{H}_{2.3}$ implies that $\left.\frac{d\tilde{\phi}_i}{d\tilde{y}_i}\right|_{y=\tilde{z}_1}$ are non-singular matrices.